1. (a) Let $f(z_0, z_1, z_2) = (z_0)^3 + (z_1)^3 + (z_2)^3$. Show that the zero set of $f$, $Z(f)$ is well-defined in and is a smooth submanifold of $\mathbb{CP}^2 = (\mathbb{C}^3 - \{0\})/\mathbb{C}^*$. 

(b) We use the notation $[z_0 : z_1 : \ldots : z_n]$ for the equivalence class of the point $(z_0, z_1, \ldots, z_n)$ in $\mathbb{CP}^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$. Consider the smooth map $\pi : Z(f) \to \mathbb{CP}^1$ given by $[z_0 : z_1 : z_2] \mapsto [z_0 : z_1]$. (Why is this well-defined?) Show that the preimage under $\pi$ of a small ball around $p \in \mathbb{CP}^1$ is either a disjoint union of three balls or, for finitely many $p$ (which/how many?), is a single ball.

(c) Using e.g. cut and paste techniques, Identify $Z(f)$ with a standard closed surface. (Bonus: Generalize to $f(z_0, z_1, z_2) = (z_0)^n + (z_1)^n + (z_2)^n$.)

2. (a) Our (current, somewhat simplistic) definition of a manifold being orientable is that it admits an atlas such that the Jacobian determinant of each transition function is everywhere positive. Show that, for $M$ any manifold, the total spaces of $T^*M$ and $TM$ are orientable.

(b) Let det : $GL_n(\mathbb{R}) \to \mathbb{R}$ be the determinant. Consider $E_{\det}$, the associated vector bundle, whose sections are tensors of type det. Show that $E_{\det}$ has a nonvanishing section, i.e. there is a nonvanishing tensor of type det, if and only if $M$ is orientable by the definition in part a.

3. (a) Consider the trivial line bundle $\mathbb{R}$ over $S^1$ and the Möbius bundle $M$ over $S^1$. Which of $\mathbb{R} \oplus \mathbb{R}$, $\mathbb{R} \oplus M$, and $M \oplus M$ are isomorphic?

(b) Consider the inclusion $\iota : \mathbb{RP}^1 \to \mathbb{RP}^2$ given by $[x_0 : x_1] \mapsto [x_0 : x_1 : 0]$ (notation as in problem 1 above). Identify the pullback $\iota^*T\mathbb{RP}^2$ with one of the bundles from part a.

4. (a) Show that there are no nonvanishing sections of $TS^2$.

(b) Show that $TS^n \oplus \mathbb{R}$ is isomorphic to the trivial bundle $\mathbb{R}^{n+1}$. 
