MATH 230A, FALL 2010, PROBLEM SET 13
DUE DECEMBER 1

1. (Chern character) Let $P$ be a principal $U(n)$-bundle over $M$. Consider $f(t,X) = \text{tr}(\exp (\frac{i}{2\pi} tX)) = \text{tr}(\sum_{k=0}^{\infty} \frac{1}{k!}(\frac{i}{2\pi} tX)^k)$, where $X \in \mathfrak{u}(n) \subset \text{Mat}_{n \times n}(\mathbb{C})$ and $t$ is a formal parameter. This is to be thought of as a formal power series in $t$ with coefficients functions of $X$ (we’ll later do away with all but finitely many terms in any given situation).

Let $f_k(X) = [t^k]f(t,X) = \frac{1}{k!}(\frac{i}{2\pi})^k \text{tr}(X^k)$, i.e. the coefficient of $t^k$ in $f(t,X)$. Note that $f_k(X)$ is a homogeneous polynomial on $\mathfrak{u}(n)$ of degree $k$. Let $ch_k(P) = W_P(f_k) \in H^{2k}_{\text{DR}}(M)$, and let $ch(P) = \sum_k ch_k(P) \in H^{\text{even}}_{\text{DR}}(M) = \oplus_k H^{2k}_{\text{DR}}(M)$.

(a) Note that $ch_0(P) = n \in H^0_{\text{DR}}(M)$, and $ch_1(P) = c_1(P)$ and, since we have $\sum_{i<j} a_i a_j = \frac{1}{2}(\sum a_i)^2 - \frac{1}{2}(\sum a_i^2)$, we have $c_2(P) = \frac{1}{2} c_1(P)^2 - c_2(P)$, so $ch_2(P) = \frac{1}{2} c_1(P)^2 - c_2(P)$.

Compute $ch_3(P)$ in terms of $c_1(P)^3$, $c_1(P)c_2(P)$, and $c_3(P)$. (Note: all products are wedge products, which are commutative since we’re in $H^{\text{even}}$.)

(b) Given a complex vector bundle with a hermitian metric $E$, we may take the associated hermitian frame bundle $P_E$, a $U(n)$ principal bundle. Then we define $ch(E) = ch(P_E)$. Show that $ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2)$. (In contrast to $c(E) = \sum_k c_k(E)$ which we saw satisfies $c(E_1 \oplus E_2) = c(E_1)c(E_2)$).

(c) The tensor product representation $\rho : U(n) \times U(m) \to U(nm)$ given by $\rho(A,B) = A \otimes B$ acting on the Lie algebras $d\rho : \mathfrak{u}(n) \oplus \mathfrak{u}(m) \to \mathfrak{u}(nm)$ is given by $d\rho(X_1, X_2) = X_1 \otimes I + I \otimes X_2$. Show that $\text{tr}(\exp(d\rho(X_1, X_2))) = \text{tr}(\exp(X_1)\text{tr}(\exp(X_2)))$. (Hint: It suffices to check on diagonal matrices since both sides are $Ad$-invariant.)

(d) Show that $ch(E_1 \otimes E_2) = ch(E_1)ch(E_2)$.

(e) Compute $c_2(E_1 \otimes E_2)$ in terms of the ranks of $E_1$ and $E_2$ and in terms of $c_1$ and $c_2$ of $E_1$ and $E_2$. (Note: This problem uses $c_k$ not $ch_k$.)

Remark: This problem shows that $ch$ is a ring map from $K(X)$, the $K$-theory of $X$, to $H^{\text{even}}_{\text{DR}}(M)$, the even (de Rham in our case) cohomology of $X$.

2. Let $P$ be a principal $G$-bundle over $M$ and let $\mathfrak{g}$ be the Lie algebra of $G$. Let $A$ be a connection on $P$.

(a) Suppose $\eta$ is a tensorial element of $\Omega^1(P, \mathfrak{g})$. (Recall that tensorial in this context means $m^*_\eta = Ad_g^{-1}\eta$ and $\iota_V \eta = 0$ for $V$ vertical.) Show that $D_A \eta = d\eta + [A, \eta]$, where $D_A \eta = (d\eta) \circ \pi_A$, where $\pi_A$ is the projection to the horizontal with kernel the vertical vectors. (Hint: Test each side on two horizontal vector fields, two vertical vector fields, and
a horizontal and vertical vector field. Another hint: Assume the horizontal vector field is $G$-invariant.)

(b) (Bonus) Suppose $\eta$ is a tensorial element of $\Omega^k(P, g)$. Show that $A_D\eta = d\eta + [A, \eta]$. (Hint: Look up the formula for $d\eta$ acting on a collection of vector fields and reason as in (a).)

(c) Give another proof of the Bianchi identity $D_AF_A = 0$ using $D_AF_A = dF_A + [A, F_A]$ and the formula for $F_A$ in terms of $A$.

(d) Show that $D_AD_A\eta = [F_A, \eta]$ for $\eta$ a tensorial element of $\Omega^k(P, g)$.

3. (Chern-Simons) Let $P$ be a principal $G$ bundle on $M$. Let $A_0$ and $A_1$ be two connections (e.g. if $P$ is trivial, we could let $A_0 = A_{\text{triv}}$) and let $\alpha = A_1 - A_0$, a tensorial element of $\Omega^1(P, g)$. Let $A_t = A_0 + t\alpha$.

(a) Show that $\frac{d}{dt}F_{A_t} = D_A\alpha$. (Hint: Use 2(a) above.)

(b) Let $f_k$ be an Ad-invariant homogeneous polynomial of degree $k$ on $g$ (or rather its associated Ad-invariant $k$-ary linear form). Let $\beta_k = k\int_0^1 f_k(\alpha \wedge (\Lambda^k F_{A_t}))dt$. Show that $d\beta_k = f_k(\Lambda^k F_{A_1}) - f_k(\Lambda^k F_{A_0})$. (Hint: Recall that $d = D_{A'}$, for any connection $A'$, on forms pulled back from the base $M$. In particular, you might choose $d = D_{A_t}$.)

(c) Suppose $P$ is the trivial principal bundle over $M$. Let $A_0 = A_{\text{triv}}$ and $A_1 = A_0 + \alpha$. (Then $d\beta_k = f_k(\Lambda^k F_{A_1})$.) Then $F_{A_0} = t\alpha + t^2\frac{1}{2}[\alpha, \alpha]$. Compute $\beta_1$, $\beta_2$, and $\beta_3$ explicitly in terms of $\alpha$ by performing the integration in (b).

(d) (Bonus) Let $Y$ be an oriented three manifold and let $P$ be a (trivializable) principal $SU(2)$ bundle over $Y$ (in fact any principal $SU(2)$ bundle over $M$ is trivializable). Let $\tau$ be a trivialization of $P$ and let $A_\tau$ be the trivial connection with respect to this trivialization. Let $A$ be a connection on $P$ which is $A_\tau + \alpha$. Let $CS_\tau(A) = \frac{1}{2\pi^2} \int_M \text{Tr}(\alpha \wedge d\alpha + \frac{3}{2} \alpha \wedge \frac{1}{2} [\alpha, \alpha])$. Show that $CS_\tau(A) - CS_{\tau'}(A) \in \mathbb{Z}$, and so we have a well defined value $CS(A) \in \mathbb{R}/\mathbb{Z}$.

(Hint: Use the fact (which we haven’t proved) that there exists an oriented four-manifold with boundary $W$ such that $\partial W = Y$ to write this difference as $c_2(P)$ for $P$ some principal $SU(2) \subset U(2)$-bundle over a closed, oriented four-manifold $M$. Use the fact (which follows with mild knowledge of homotopy theory from our calculations on $S^4$) that in this case $c_2(P)$ is an integer.)

Remark: Note that (b) gives an explicit proof that the difference in the curvature of two connections on the same principal bundle is exact. Part (b) was known quite early, but unless my history is wrong, part (c) was not tried (or at least used) until the paper of Chern and Simons from 1974. The functional $CS$ is important in various (related) contexts in physics, knot theory, and 3-and 4-manifold topology. Its critical points are the flat connections (try it!).