1. **Hodge star**

Let $V$ be a real, dimension $n$, vector space with inner product.

(a) Show that there is an inner product on $\wedge^k V = \bigoplus_{k=0}^n \wedge^k V$ such that the $\wedge^k V$ are orthogonal for different $k$ and such that $\langle v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_k \rangle = \det(\langle v_i, w_j \rangle)$.

Now suppose $V$ has an orientation; equivalently a choice of “positive” component of $\wedge^n V \setminus \{0\}$. We give an isomorphism $\phi : \wedge^n V \to \mathbb{R}$ by identifying 1 with the positive element of norm one (with respect to the inner product above).

(b) For $\eta \in \wedge^k V$, define $*\eta \in \wedge^{n-k} V$ such that $\phi(\omega \wedge *\eta) = \langle \omega, \eta \rangle$ for $\omega \in \wedge^k V$. Show that $** : \wedge^k V \to \wedge^k V$ is multiplication by $(-1)^{k(n-k)}$.

Now suppose $M$ is an oriented compact Riemannian manifold without boundary. Here Riemannian means we have chosen a Riemannian metric, i.e. an symmetric, positive definite bilinear form $g$ on each tangent space $TM$ which varies smoothly. (E.g. in coordinates the inner product $g(v, w) = v^T A w$ is given by a matrix $A$ whose coefficients vary smoothly.) By the above we get inner products on exterior powers of $T^*M$ (or equivalently $TM$) and the Hodge star $* : \wedge^k T^* M \to \wedge^{n-k} T^* M$ (equivalently for $TM$).

(c) We have $d : \Omega^k(M) \to \Omega^{k+1}(M)$. Define $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ as $\delta = \pm \ast d \ast$ (I'll let you determine this sign below). Define an inner product on $\Omega^k(M)$ via $\langle \omega, \eta \rangle := \int_M \langle \omega_p, \eta_p \rangle \, dvol = \int_M \omega \wedge *\eta$. (Here $dvol$ is the $n$-form which pointwise is of norm one and agrees with the orientation, i.e. is in the “positive” component of $\wedge^n T^* M_p$.) Show that $\langle d\omega, \eta \rangle = \langle \omega, \delta \eta \rangle$ for an appropriate sign choice in the definition of $\delta$ (what is that choice?).

We make this definition of $\delta$ (with the sign choice you found) whether $M$ is compact without boundary or not. Sometimes the notation $d^*$ is used, which is appropriate as this is the $(L^2)$- adjoint of $d$ when $M$ is compact without boundary.

(d) Given the metric $g$, we have get an isomorphism between $TM$ and $T^* M$ which we’ll call $\theta_g : TM \to T^* M$. Here $\theta_g(v)$ is such that $g(v, u) = [\theta_g(v)](u)$. Let $V$ be a vector field on $M$. Define $\text{div}_g V = \delta(\theta_g V)$. For dim $M = 3$, define $\text{curl}_g V = \theta_g^{-1}(d(\theta_g V))$. Show that for $\mathbb{R}^n$ with the standard inner product on $T \mathbb{R}^n = \mathbb{R}^n$, these are the usual divergence and curl.

2. **Electromagnetism and $U(1)$ gauge theory**

Recall Maxwell’s equations for the electric field $E$ and magnetic field $B$, time-dependent vector fields on $\mathbb{R}^3$ (i.e. really they are vector fields on $\mathbb{R}^4$ which are tangent to $\{pt\} \times \mathbb{R}^3$):

\begin{align*}
(1) \quad \text{div} \ E & = \rho \\
(2) \quad \text{div} \ B & = 0 \\
(3) \quad \text{curl} \ E & = -\partial_t B \\
(4) \quad \text{curl} \ B & = J + \partial_t E
\end{align*}
(in units where \(\epsilon_0 = \mu_0 = 1\)). Here \(\rho\), a (time-dependent) function, is the charge density and \(J\), a (time-dependent) vector field, is the current.

(a) Let \(E = \theta_y E\) and \(B = \theta_y B\). Show there is a (time-dependent) one-form \(A\) such that \(*B = dA\). (Hint: Use the Poincaré lemma. If we haven’t covered this lemma in class by the time you do this problem, look it up in chapter 4 of the book.)

(b) Show there is a (time-dependent) function \(\phi\) such that \(-d \phi = E + \partial_t A\).

(c) Let \(F\) be the two-form on \(\mathbb{R}^4\) given by \(F = -dt \wedge E + *_3 B\). (Here \(*_3\) means that we take the Hodge star of \(B\) on \(\mathbb{R}^3\) at each time to get a time-dependent two-form, which we then think of as a two-form on \(\mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4\). This is the same as the \(*\) that appeared in (a) where we were thinking of \(B\) as time-dependent.) Let \(J = \rho dt + \theta_y J\), a one-form on \(\mathbb{R}^4\).

Show that Maxwell’s equations are equivalent to the two equations \(dF = 0\) and \(\delta F = J\).

(d) Let \(\alpha\) be the one-form \(-\phi dt + A\) on \(\mathbb{R}^4\). Show that \(d\alpha = F\) and that Maxwell’s equations are equivalent to \(\delta d\alpha = J\).

(e) Show that Maxwell’s equations are preserved by adding \(df\) to \(\alpha\).

Remark: The form \(\alpha\), modulo terms of the form \(df\) (sort of like its homology class, except that \(d\alpha\) need not be zero) is in fact physical, by the Aharonov-Bohm effect, which states that if \(\alpha\) has a nonzero integral over the image of a circle in a subset of \(\mathbb{R}^4\) where \(F = 0\) (and thus \(E = B = 0\)) [the subset of \(\mathbb{R}^n\) will necessarily be non-simply-connected], then this has a (quantum mechanical) effect on certain particles travelling in that region (by changing the phase of their wave functions, which may alter interference patterns, e.g.).