1. (a) Let \( \omega \in \Lambda^2(\mathbb{R}^n \oplus \mathbb{R}^n) \) with \( \omega = \sum_i e_i \wedge f_i \), where the \( e_i \) are a basis for the first \( \mathbb{R}^n \) factor and the \( f_i \) are a basis for the second \( \mathbb{R}^n \) factor. What multiple of \( e_1 \wedge e_2 \ldots \wedge e_n \wedge f_1 \wedge f_2 \ldots \wedge f_n \) is \( \wedge^n \omega = \omega \wedge \omega \ldots \wedge \omega \in \Lambda^{2n}(\mathbb{R}^n \oplus \mathbb{R}^n) \) (with \( n \) copies of \( \omega \)).

(b) With the convention that \( \ell_1 \wedge \ell_2 = \ell_1 \otimes \ell_2 - \ell_2 \otimes \ell_1 \), an element of \( \Lambda^2(V^*) \) corresponds to a skew-symmetric bilinear form: \( (\ell_1 \wedge \ell_2)(v \otimes w) = (\ell_1 \otimes \ell_2)(v \otimes w) - (\ell_2 \otimes \ell_1)(v \otimes w) = \ell_1(v)\ell_2(w) - \ell_2(v)\ell_1(w) \).

Consider \( \mathbb{R}^n \times \mathbb{R}^n \) and let \( v_i \) be a basis for the first factor and \( w_i \) be a basis for the second. Let \( e_i = v_i^* \), a dual basis for \( \mathbb{R}^n \), and \( f_i = w_i^* \) a dual basis, and consider \( \omega \in \Lambda^2((\mathbb{R}^n)^* \oplus (\mathbb{R}^n)^*) \) as above, i.e. \( \omega = \sum_i v_i^* \wedge w_i^* \).

Given \( v \in \mathbb{R}^n \) and \( w \in \mathbb{R}^n \) we have \( \omega(v, w) \), (skew-symmetric and) bilinear. Hence there exists a unique \( A \in \text{Mat}_{2n \times 2n}(\mathbb{R}) \) such that \( \omega(v, w) = v^T A w \). Compute the matrix \( A \). (Express it with \( n \times n \) blocks perhaps.)

2. Suppose \( \alpha \in \Omega^1(S^1) \) and \( \int_{S^1} \alpha = 0 \). Construct a function \( f : S^1 \to \mathbb{R} \) such that \( \alpha = df := f'(t)dt \) (i.e. this is true for any choice of coordinate \( t \) on any part of \( S^1 \)).

Hint: Use a partition of unity.

3. **Curl and Stokes’s Theorem**

(a) Let \( X \) be a vector field on \( \mathbb{R}^3 \). Let \( \mathbf{u} \) be a unit vector and let \( \mathbf{v} \) and \( \mathbf{w} \) complete \( \mathbf{u} \) to an oriented, orthonormal basis. Show that \( \text{curl}(X)(p) \cdot \mathbf{u} \) is equal to the two-dimensional curl of the vector field on \( \mathbb{R}^2 \) (with coordinates \( s \) and \( t \)) with components \( (X(p + sv + tw) \cdot v, X(p + sv + tw) \cdot \mathbf{w}) \). That is, it equals \( \frac{\partial X(p + sv + tw) \cdot \mathbf{w}}{\partial s} - \frac{\partial X(p + sv + tw) \cdot \mathbf{v}}{\partial t} \).

(b) Use Green’s theorem to prove Stokes’s theorem (non-general version) using part (a). These are stated below.

**Green’s Theorem:** Let \( D \subset \mathbb{R}^2 \) be a compact, smooth two-manifold with boundary \( \partial D \subset \mathbb{R}^2 \) (a smooth one-manifold). Given \( (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2 \), we have
\[
\int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy = \int_A (P, Q) \cdot r'(t) dt,
\]
where \( r : [a, b] \to \partial D \) parametrizes the boundary of \( D \) counterclockwise.

**Stokes’s Theorem, non-general version:** Let \( S \subset \mathbb{R}^3 \) be a compact, smooth two-manifold with boundary \( \partial S \subset \mathbb{R}^3 \) (a smooth one-manifold). Given \( X = (P, Q, R) : \mathbb{R}^3 \to \mathbb{R}^3 \), we have \( \int_A (P, Q, R) \cdot r'(t) dt = \int_{\partial S} \text{curl}(X) \cdot \mathbf{n} |f_s(s, t) \times f_t(s, t)| ds \, dt \). Here \( r : [a, b] \to \partial S \) is a (boundary-oriented) parametrization, and \( f : R \to S \) is an oriented parametrization, with \( R \subset \mathbb{R}^2 \) (with coordinates \( s, t \)). Here \( f_s \) and \( f_t \) are the partial derivatives (each with three coordinates) and \( f_s \times f_t \) is the cross product. Also, \( \mathbf{n} \) is a unit normal vector compatible with orientations.

**Remark:** What is going on here and the relationship between the two theorems and the two and three dimensional curls will become clearer as we cover the exterior derivative and the general Stokes theorem. Also, compare part (a) to the “gradient vector dot unit vector equals directional derivative” concept.