MATH 132, SPRING 2013, PROBLEM SET 4
DUE MARCH 1

1. General linear position
(a) Consider triples of points \( u, v, w \in \mathbb{R}^2 \), which we may consider as single points \((u, v, w)\) \(\in \mathbb{R}^6\). Show that for almost every \((u, v, w)\) \(\in \mathbb{R}^6\), the points \(u, v, w\) are not collinear.

(b) Consider \(n\)-tuples of points in \(\mathbb{R}^2\), thought of as single points in \(\mathbb{R}^{2n}\). Show that for almost every such, no three of the points are collinear.

(c) Consider \(n\)-tuples of points in \(\mathbb{R}^k\), thought of as single points in \(\mathbb{R}^{kn}\). Show that for almost every such, for all \(d \leq k + 1\) we have that no \(d\) of the points lie on the same \((d - 2)\)-dimensional plane.

2. Hausdorff dimension and Sard’s theorem
We say that a set \(A \subset \mathbb{R}^n\) has \(d\)-dimensional Hausdorff measure zero if for all \(\epsilon > 0\) there exists a covering of \(A\) by countably many cubes \(S_i\) with side lengths \(s_i\) such that \(\sum_i (s_i)^d < \epsilon\). The Hausdorff dimension of \(A\) is then defined to be the infimum over \(d \in \mathbb{R}\) such that \(A\) has \(d\)-dimensional Hausdorff measure zero.\(^1\)

(a) Show that a \(k\)-dimensional cube included into \(\mathbb{R}^n\) has Hausdorff dimension \(k\).

(b) Suppose \(f : \mathbb{R}^1 \to \mathbb{R}^1\). Let \(C\) be the set of critical points of \(f\). Show that \(f(C)\) has Hausdorff dimension at most zero.

Remark: If you trace through our proof of Sard’s theorem, you should be able to prove that if \(f : \mathbb{R}^n \to \mathbb{R}^m\) has critical points \(C\), then \(f(C)\) has Hausdorff dimension at most \(\min(n - 1, m - 1)\): the \(n - 1\) comes from reasoning as in (b) and the \(m - 1\) comes from Fubini’s theorem for Hausdorff measure and the changing coordinates tricks. There’s also a generalization that if \(R_k\) is the set of points \(x\) such that \(df_x\) has rank at most \(k\) (this is different from our \(C_k\)), then \(f(R_k)\) has Hausdorff measure at most \(k\); note this implies the statement in the preceding sentence.

3. [GP, 2.3.8]
Suppose \(m > 1\). Let \(f : \mathbb{R}^n \to \mathbb{R}^m\) and let \(K \subset \mathbb{R}^n\) be compact. Show that for any \(\epsilon > 0\) there exists \(g : \mathbb{R}^n \to \mathbb{R}^m\) such that \(|f(x) - g(x)| < \epsilon\) for \(x \in K\) and \(dg_x \neq 0\) for all \(x \in \mathbb{R}^n\).

Hint: Consider \(f + Ax\) for \(A \in \text{Mat}_{m \times n}\). Define \(F : \mathbb{R}^n \times \text{Mat}_{m \times n} \to \text{Mat}_{m \times n}\) by \(F(x, A) = df_x + A\) and apply the regular value theorem.

Remark: This doesn’t get rid of critical points, just points where \(df_x = 0\). Can you improve the result to a better lower bound on the rank of \(dg_x\)?

4. Let \(f : X \to \mathbb{R}^2\).
(a) Show that for almost every \(c \in \mathbb{R}\), we have that \(f^{-1}((c) \times \mathbb{R})\) is a smooth submanifold of \(X\).

\(^1\)This is at least zero if \(A\) is nonempty. If \(A\) is empty, you may say \(A\) has Hausdorff dimension \(-\infty\) if you wish.
(b) Let $\mathcal{L} = \{\text{lines in } \mathbb{R}^2\}$. Consider the 2-to-1 map $f : S^1 \times \mathbb{R} \to \mathcal{L}$ given by $(\theta, x) \mapsto L_{\theta, x} = \{t(\cos \theta, \sin \theta) + x(-\sin \theta, \cos \theta) : t \in \mathbb{R}\}$.

Show that for almost every $(\theta, x) \in S^1 \times \mathbb{R}$, we have that $f^{-1}(L_{\theta, x})$ is a smooth submanifold of $X$.

Remark: The set of lines in $\mathbb{R}^2$ is a Möbius band.

5. Almost every projection of a knot is a knot diagram (part one)

Let $f : S^1 \to \mathbb{R}^3$ be a knot, i.e. a smooth embedding (i.e. 1-to-1 and with injective derivative, since $S^1$ is compact).

(a) Given an element $v \in S^2$ we have the orthogonal projection $\pi_v : \mathbb{R}^3 \to P_v$ to the plane $P_v = \{v\}^\perp$. Show that for almost every $v \in S^2$, we have that $f_v := \pi_v \circ f : S^1 \to P_v$ is an immersion.

Hint: What does it mean for $v$ not to work? Consider the map $S^1 \to S^2$ taking $\theta \in S^1$ to $\frac{\partial f}{\partial \theta} |_{\partial f/\partial \theta} \in S^2$.

(b) Let $X = S^1 \times S^1 - \Delta$ where $\Delta = \{(x, x) : x \in S^1\}$. This is an open subset of $S^1 \times S^1$ and hence a manifold. Consider the map $G : X \to S^2$ such that

$$G(x, y) = \frac{f(x) - f(y)}{|f(x) - f(y)|}$$

Show that if $v \in S^2$ is a regular value of $G$ and $f_v$ is an immersion, then $f_v = \pi_v \circ f$ has transverse crossings. That is, $f_v(x) = f_v(y)$ implies that $\frac{\partial f_v}{\partial \theta}(x)$ and $\frac{\partial f_v}{\partial \theta}(y)$ are linearly independent.

Remark 1: A knot diagram is an immersion from $S^1$ to $\mathbb{R}^2$ with all crossings being transverse double points (i.e. only two preimages at most) together with the data of which strand of the knot is “on top” at each crossing.

Remark 2: Note that $v$ and $-v$ in $S^2$ give the same projection, so really the space of projections is something called real projective two space, $\mathbb{RP}^2$. This is covered two-to-one by $S^3$ and is the cross cap surface from the first lecture. Note that in a previous problem set we saw $SO(3)$ is covered two-to-one by $S^3$ with preimages given by pairs of opposite points, so $SO(3)$ is diffeomorphic to $\mathbb{RP}^3$. 