MATH 132, SPRING 2014, PROBLEM SET 6
DUE APRIL 24

1. Families of Morse functions

(a) [GP 1.7.16-18] Show that for $X$ compact, Morse functions $f : X \rightarrow \mathbb{R}$ form a stable class of maps. That is, if $F : X \times I \rightarrow \mathbb{R}$ is smooth and $f_0(x) = F(x, 0)$ is Morse, then so is $f_t(x) = F(x, t)$ for small $t$.

Hint: Show that, in a chart, $f$ is Morse if and only if

$$(\det (\text{Hess}(f)))^2 + \sum_i \left( \frac{df}{dx_i} \right)^2 > 0.$$ 

(b) Suppose $F(x, t) : X \times I \rightarrow \mathbb{R}$ is a homotopy of Morse functions. That is, $f_t : X \rightarrow \mathbb{R}$ is Morse for every $t$. Show that the set $C = \{(x, t) \in X \times I : d(f_t)_x = 0\}$ forms a closed, smooth submanifold of dimension one of $X \times I$.

(c) Let $\pi : X \times I \rightarrow I$. Show that $d(\pi|_C)(x, t) : TC_{(x, t)} \rightarrow TI_t$ is surjective.

(d) Show that if $X$ is compact, there is no homotopy of Morse functions between two Morse functions with different numbers of critical points.

2. [GP, 2.4.5,9,10]

(a) A contractible manifold $Y$ is one for which the identity map is homotopic to a constant map. Suppose $Y$ is contractible (with dim$(Y) > 0$) and $X \subset Y$ compact submanifold, $Z \subset Y$ closed submanifold, $0 < \text{dim}(X)$ and $\text{dim}(X) + \text{dim}(Z) = \text{dim}(Y)$. Show that $I_2(X, Z) = 0$.

(b) Conclude that a compact manifold is not contractible.

(c) Suppose $X$ compact with $0 < \text{dim}(X) < k$ and $f : X \rightarrow S^k$. Suppose $Z \subset S^k$ a closed submanifold with $\text{dim}(X) + \text{dim}(Z) = k$. Show that $I_2(X, Z) = 0$.

(d) Let $Y = X \times Z$. Show that $I_2(X \times \{0\}, \{0\} \times Z) = 1$.

(e) Conclude $S^k$ and $T^k$ are not diffeomorphic.

3. (a) Let $X$ be a compact $k$-manifold. Construct a map from $X$ to $S^k$ of odd degree.

(b) Suppose $f : X \rightarrow Y$ is an odd degree map of compact manifolds of the same dimension. Let $Z_1$ and $Z_2$ be two transversally intersecting closed submanifolds of $Y$ of complementary dimension with $I_2(Z_1, Z_2) = 1$. Show that there exists $g : X \rightarrow Y$ homotopic to $f$ and transversal to $Z_1$ and to $Z_2$ and to $Z_1 \cap Z_2$. Show that for such a $g$ we have $I_2(g^{-1}(Z_1), g^{-1}(Z_2)) = 1$.

(c) Conclude from this and problem 2 that there is no odd degree map from $S^k$ to $T^k$. 

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