Almost every projection of a knot gives a knot diagram

Let \( f : S^1 \to \mathbb{R}^3 \) be a knot, i.e. a smooth embedding (which here is equivalent to being 1-to-1 and an immersion, since \( S^1 \) is compact).

We’ve seen that, for almost every \( v \in \mathbb{R}^3 \), or for almost every \( v \in S^2 \subset \mathbb{R}^3 \), we have that \( \pi_v \circ f : S^1 \to \{v\}^\perp \cong \mathbb{R}^2 \) is an immersion. Call this map \( f_v = \pi_v \circ f \).

A **knot diagram** is: An immersion from \( S^1 \) to \( \mathbb{R}^2 \) with all crossings (i.e. points that are the image of more than one point) being transverse double points, together with the data of which strand is “on top” at each crossing. A **transverse double point** is a crossing whose preimage is only two points, and the images of neighborhoods of those two points meet transversally at the crossing.

We show in this problem that for almost every \( v \), we have that \( f_v \) gives a knot diagram.

(a) Let \( X = S^1 \times S^1 - \Delta \), where \( \Delta = \{(x, x) : x \in S^1\} \). Consider the map \( G : X \to S^2 \)

\[
G(x, y) = \frac{f(x) - f(y)}{|f(x) - f(y)|}.
\]

Suppose \( f_v(a) = f_v(b) \). Show that \( dG/dx(a, b) \) is a nonzero multiple of \( d(f_v)/dt(a) \) and that \( dG/dy(a, b) \) is a nonzero multiple of \( d(f_v)/dt(b) \).

**Hint:** Don’t be afraid to get your hands dirty and take derivatives and use the chain rule. Alternatively, there is a clever symmetry argument that obviates having to do much calculation here.

(b) Show that if \( v \in S^2 \) is a regular value of \( G \) and \( f_v = \pi_v \circ f \) is an immersion, then \( f_v \) has **transverse crossings**. That is, \( f_v(x) = f_v(y) \) implies that \( d(f_v)_x((TS^1)_x) \) and \( d(f_v)_y((TS^1)_y) \) are transverse.

(c) Suppose \( f_v \) is an immersion and \( v \) is a regular value of \( G \). Show that \( v \) has an open neighborhood \( U \) consisting of regular values for \( G \) such that \( \#G^{-1}(w) = \#G^{-1}(v) \) for all \( w \in U \).

**Hint:** The Stack of Records theorem doesn’t apply due to the lack of compactness of \( X \). Make it work anyway, using the fact that there are only two accumulation points of \( G(x, y) \) as \( x \to y \) (the unit tangent vector to the knot and its opposite).

(d) Let \( B(x, y, z, t) = [f(x) - f(y)] + t[f(y) - f(z)] = f(x) + (t - 1)f(y) - tf(z) \), with \( B : S^1 \times S^1 \times S^1 \times \mathbb{R} \to \mathbb{R}^3 \). Show that if we have \( x, y, z \) all different and \( B(x, y, z, t) = 0 \) and \( G(x, y) = \pm G(y, z) = v \) is a regular value for \( G \), then \( (x, y, z, t) \) is a regular point for \( B \).

(e) Conclude that for almost every \( w \in U \) we have that \( f_w^{-1}(p) \) consists of at most two points for all \( p \in \{w\}^\perp \).

(f) Conclude that for almost every \( v \in S^2 \), we have that \( f_v : S^1 \to \{v\}^\perp \cong \mathbb{R}^2 \) is an immersion whose crossings are transverse double points. That is, \( f_v \), together with the data of which strand is on top at each crossing, gives a knot diagram.