1. (a) Define immersion.
A smooth map \( f : X \to Y \) is an immersion if \( df_x : TX_x \to TY_{f(x)} \) is injective for all \( x \in X \).

(b) Define what it means for a property of smooth maps to be stable.
A property of smooth maps, say from \( X \) to \( Y \), is stable if for any smooth map \( F : X \times I \to Y \) such that \( F(\cdot, 0) : X \to Y \) has that property, there exists \( \epsilon > 0 \) such that that \( F(\cdot, t) \) has that property for \( 0 \leq t < \epsilon \).

(c) Suppose \( X \) is compact. Show that the property of being an immersion from \( X \) to \( Y \) is a stable property among maps from \( X \) to \( Y \).
Let \( f_t(x) = F(x, t) \) for \( F : X \times I \to Y \) a homotopy with \( f_0 : X \to Y \) an immersion. Let \( X \) and \( Y \) be \( k \) and \( \ell \) dimensional, respectively.
First, we claim that given \( (p, 0) \in X \times I \), there is an open set \( U_p \subset X \times I \) containing it such that \( (df_t)_x \) is injective for \( (x, t) \in U_p \). To get such an open set, we work in a chart \( \phi : V \to X \) with \( \phi(0) = p \) (and \( 0 \in V \)) and give a map \( g : V \times I \to \mathbb{R}^k \) with \( g(x, t) \) giving the determinants of \( k \times k \) minors of \( (df_t)_x \). A linear map from \( \mathbb{R}^k \) to \( \mathbb{R}^\ell \) is injective if and only if some \( k \times k \) minor has nonzero determinant. Then we let \( U_p \) be the image under \( \phi \) of \( V - g^{-1}(0) \).
Next, we claim that there exists \( \epsilon > 0 \) such that \( X \times [0, \epsilon) \subset \bigcup_{p \in X} U_p \). To see this, shrink each \( U_p \) to be a product open set of the form \( V_p \times [0, \epsilon_p) \) with \( V_p \subset X \) open containing \( p \). Then notice only finitely many \( V_p \) are required to cover \( X \) by compactness. Let \( \epsilon \) be the minimum of the \( \epsilon_p \) for the finite collection of \( V_p \) that cover.

2. Determine whether the equation(s) smoothly cut(s) out a manifold, and what the dimension of the manifold is if so:
(a) \( xyz = 1 \) in \( \mathbb{R}^3 \)
Let \( g(x, y, z) = xyz \). Then \( dg_{(x,y,z)} = (yz, xz, xy) \). This is only zero if at least two of the \( x, y, z \) are zero. But none of \( x, y, z \) can be zero on \( g^{-1}(1) \). Hence \( dg_p \) is nonzero, hence surjective to \( \mathbb{R} \), for all solutions to \( xyz = 1 \). Hence this is a smooth manifold of dimension \( 3 - 1 = 2 \).
(b) \( x^3 - y^2 = 0 \) in \( \mathbb{R}^2 \)
Let \( g(x, y) = x^3 - y^2 \). Then \( dg_{(x, y)} = (3x^2, -2y) \). This is zero, hence not surjective, for \( (x, y) = (0, 0) \). Hence the solutions to the equation are not smoothly cut out.
(c) \( x^2 + y^2 + z^2 = 1 \) and \( x + y + z = 1 \) in \( \mathbb{R}^3 \)
Letting \( g = x^2 + y^2 + z^2 \) we have \( dg = (2x, 2y, 2z) \). For the second, let \( h = x + y + z \) and \( dh = (1, 1, 1) \). We have that \( dg \) and \( dh \) are linearly independent on solutions to the equations
if and only if the map \( f = (g, h) : \mathbb{R}^3 \to \mathbb{R}^2 \) has \( df \) surjective on preimages of \((1, 1)\). They are linearly dependent if and only if \( x = y = z \). Plugging in, we see that for the latter each would have to be 1/3 while for the former each would have to be \( \pm \frac{1}{\sqrt{3}} \). Hence they are linearly independent at all solutions to \( f = (1, 1) \) and hence the solutions are smoothly cut out.

3. We say two maps \( f : X \to Z \) and \( g : Y \to Z \) are transversal if whenever \( f(x) = g(y) \) we have \( \text{im}(df_x) + \text{im}(dg_y) = TZ_{f(x)} \). Show that \( f \) and \( g \) are transversal if and only if \( f \times g \) and \( \Delta \) are transversal, where \( f \times g : X \times Y \to Z \times Z \) is given by \((f \times g)(x, y) = (f(x), g(y))\) and \( \Delta = \{(z, z) : z \in Z\} \subset Z \times Z \).

Step 0: We have that \( T\Delta_{(x, z)} = \{(w, w) \in TZ_x \oplus TZ_z : w \in TZ_z\} \) because the map \( Z \to \Delta \) with \( z \mapsto (z, z) \) and projection to the first factor from \( \Delta \to Z \) are inverses and hence diffeomorphisms, and the derivative of the first takes \( TZ \) to the aforementioned set.

Step 1: \( f \times g \) and \( \Delta \) transversal implies \( f \) and \( g \) are transversal. We need to show that at a point \( z = f(x) = g(y) \) and a vector \( w \in TZ_z \) we have \( u \in TX_x \) and \( v \in TY_y \) such that \( df_x(u) + dg_y(v) = w \). To get this, find \((u', v') \in T(X \times Y)_{(x, y)}\) and \( a \in TZ_z \) such that \((a, a) + d(f \times g)(x, y)(u', v') = (w, 0)\) by transversality of \( f \times g \) and \( \Delta \). Then \( df_x(u') - dg_y(v') = w \). Now let \( u = u' \) and \( v = -v' \).

Step 2: \( f \) and \( g \) are transversal implies \( f \times g \) and \( \Delta \) transversal. We need to show that at a point \((z, z)\) with \( z = f(x) = g(y) \) and given \((a, b) \in TZ_z\) we have \( u \in TX_x \) and \( v \in TY_y \) and \( w \in TZ_z \) such that \((df_x(u), dg_y(v)) + (w, w) = (a, b)\). To get this, let \( u' \) and \( v' \) be such that \( df_x(u') + dg_y(v') = a - b \) by transversality of \( f \) and \( g \). Then let \( u = u' \) and \( v = -v' \) and \( w = b + dg_y(v') \).

4. Let \( X \subset \mathbb{R}^n \) be a compact submanifold of dimension \( n - 1 \) not containing the origin. Show that for almost every \( v \in S^{n-1} \), the ray \( R_v = \{tv : t \in \mathbb{R}, t \geq 0\} \) intersects \( X \) in only finitely many points.

Since \( X \) does not contain the origin, we have a map \( f : X \to S^{n-1} \) given by \( p \mapsto \frac{p}{\|p\|} \). Almost every value in \( S^{n-1} \) is regular for \( f \) by Sard’s theorem. At a regular value \( v \), we claim there are only finitely many preimages. At each preimage point, \( df \) is a local diffeomorphism. Hence there is an open set \( U \) around the preimage point such that \( f^{-1}(v) \cap U \) is only one point. Also, \( f^{-1}(v) \) is closed in a compact space \( X \), hence compact. If \( f^{-1}(v) \) were infinite, it would have an accumulation point, but this is impossible because the points are isolated (in their open sets with only one point in \( f^{-1}(v) \) in them).