1. Let $T$ be an endomorphism of a vector space $V$ of dimension $d$ (finite) over an algebraically closed field $F$. Let $n_{\lambda,k}$ denote the number of Jordan blocks of $T$ with eigenvalue $\lambda$ and size $k$ when $T$ is put in Jordan form. (Such a block is a $k \times k$ matrix with $\lambda$ on the diagonal and ones just above the diagonal.)

Give a formula for $n_{\lambda,k}$ in terms of the numbers $m_{\lambda,\ell} = \dim[\ker((T - \lambda I)^\ell)]$.

Hint: To start off, show that $n_{\lambda,d} = m_{\lambda,d} - m_{\lambda,d-1}$ (we more or less showed this in class). Now how about $n_{\lambda,(d-1)}$?

2. [Artin 2011 5.1.1] Determine the matrices that represent the following rotations of $\mathbb{R}^3$

(a) angle $\theta$ counterclockwise around the axis in the direction of $e_2$

(b) angle $2\pi/3$ counterclockwise around the axis in the direction of $(1,1,1)^T$ (written as a transpose only because we’re thinking of it as a column vector).

(c) angle $\pi/2$, counterclockwise in the direction of $(1,1,0)^T$.

3. [Artin 2011 6.3.6]

(a) Let $s$ be the rotation of the plane with angle $\pi/2$ counterclockwise about the point $(1,1)^T$. Write the formula for $s$ as a product $\tau_v \rho_\theta$.

(b) Let $s$ denote the reflection of the plane about the axis $x = 1$. Find an isometry $g$ such that $grg^{-1} = s$, where $r$ is reflection about the $x$-axis ($y = 0$). Write $s$ in the form $\tau_v \rho_\theta r$.

4. [cf. Artin 2011 6.5.10]

(a) Let $f$ and $g$ be rotation of the plane about distinct points, with arbitrary nonzero angles of rotation $\theta$ and $\phi$. Prove that $f \circ g \neq g \circ f$.

(b) Show that the group generated by $f$ and $g$ contains a nonzero translation. (That is, there is a nonzero translation that can be written as a composition of some number of copies of $f$ and $g$ and their inverses in some order.)

Hints: 1) Recall that rotation by $\theta$ counterclockwise about the point $v$ is given by $\tau_v \rho_\theta \tau_v$. 2) Consider the homomorphism $\text{Isom}(\mathbb{R}^2) \rightarrow O(2)$ coming from quotienting by translations. (Use one or both hints.)

5. Let $G$ be the group of symmetries$^1$ of the pattern $\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv\equiv$ (continued infinitely to the left and right)

(a) [cf. Artin 2011 6.5.6] The subgroup of translations in $G$, call it $G_T$, is isomorphic to $\mathbb{Z}$ (e.g., generated by translating two character-widths to the right). Identify the quotient group $\overline{G} = G/G_T$ (called the point group in the text). What is the index of $G_T$ in $G$?

(b) For each element of $\overline{G}$, write down (in $\tau_v \rho_\theta^k$ form) an element of $G$ which maps to it under the projection to the quotient $G \rightarrow G/G_T$.

(c) Prove or disprove: $G$ is isomorphic to some semidirect product of $\overline{G}$ and $\mathbb{Z}$ (in some order).

$^1$That is, isometries of the plane taking the pattern to itself.