1. Abel’s Limit Theorem
   (a) Let \( f(z) = \sum_{k=0}^{\infty} c_k z^k \) be a power series with radius of convergence \( R = 1 \). Suppose further that \( \sum_{k=0}^{\infty} c_k \) exists. Let \( s_n = \sum_{k=0}^{n} c_k \). Show that \( \sum_{n=0}^{\infty} s_n z^n \) converges for \( |z| < 1 \) and that \( f(z) = (1 - z) \sum_{n=0}^{\infty} s_n z^n \) for \( |z| < 1 \).
   (b) Show that \( \lim_{x \to 1} f(x) = \sum_{k=0}^{\infty} c_k \). (That is, we approach 1 among real \( x \) from the left.)
   \textit{Hint:} Show that it suffices to consider \( c_k \) such that \( \sum_{k=0}^{\infty} c_k = 0 \).
   \textit{Remark:} The limit in (b) may more generally be taken among complex \( z \) as long as \( z \) approaches 1 among the set of \( z \) such that \( |1 - z| \leq C(1 - |z|) \) for some constant \( C > 0 \).

2. Changing the center of a power series
   (a) Let \( f(z) = \sum_{k=0}^{\infty} c_k z^k \) be a power series with radius of convergence \( R > 0 \). Let \( w \in \mathbb{C} \) with \( |w| < R \). Show that \( f(z) \) admits a power series expansion centered on \( w \) of radius at least \( R - |w| \). Namely, show that there exist \( b_n \) such that \( f(z) = \sum_{n=0}^{\infty} b_n (z - w)^n \) with \( \lim_{n \to \infty} |b_n|^{1/n} \leq \frac{1}{R - |w|} \) (this is intended to be 0 when \( R = \infty \)).
   \textit{Hint:} Don’t try to compute \( \lim |b_n|^{1/n} \) for your favorite \( b_n \). Instead, use the fact that a double summation can be rearranged if either: 1) The terms are all positive real numbers or zero or 2) The double summation is absolutely convergent (i.e. the (double) summation of the absolute values of the terms is convergent). [There’s no need to prove this property of double summations.]
   (b) Under the same conditions, give an expression for \( f^{(n)}(w) \) (the \( n \)-th derivative of \( f \) evaluated at \( w \)) in terms of the \( c_k \).

3. Summation by parts
   (a) Given sequences \( a_k \) and \( b_k \), show that
   \[ \sum_{n=1}^{N} a_n b_n = a_N \sum_{n=1}^{N} b_n + \sum_{n=1}^{N-1} \left( a_n - a_{n+1} \right) \sum_{k=1}^{n} b_k \].
   (b) Show that \( f(z) = \sum_{n=1}^{\infty} \frac{a_n}{n} \) converges for all \( z \) such that \( |z| \leq 1 \) and \( z \neq 1 \).
   (c) Show that \( f(z) \) from part (b) is a continuous map from \( \mathbb{D} - \{1\} \to \mathbb{C} \), where \( \mathbb{D} \) is the closed unit disk in \( \mathbb{C} \).
   \textit{Hint:} For part (c), don’t apply problem 1. Instead, show uniform convergence on closed subsets of \( \mathbb{D} \) not containing 1.
   \textit{Remark:} For real \( z \), the function \( f(z) \) equals \(-\log(1 - z)\), and for complex \( z \), this is the principal branch of that function. Notice that the function itself is defined and holomorphic outside the radius of convergence of its Taylor series, just not in all directions.
4. Cauchy-Riemann in polar coordinates
   (a) Show that in polar coordinates \( r, \theta \), away from \( z = 0 \), the Cauchy-Riemann equations are equivalent to
   \[
   \begin{align*}
   \frac{\partial u}{\partial r} &= \frac{1}{r} \frac{\partial v}{\partial \theta} \\
   \frac{\partial v}{\partial r} &= -\frac{1}{r} \frac{\partial u}{\partial \theta}
   \end{align*}
   \]
   (b) Show directly, by using the equations found in (a) that (any branch of) the complex logarithm function \( \log(z) = \log(r) + i\theta \), where \( z = re^{i\theta} = r(\cos \theta + i \sin \theta) \), is holomorphic.

5. Harmonic functions
   (a) Show that if \( f(z) = u(x,y) + iv(x,y) \) is holomorphic and \( C^2 \) (twice continuously differentiable), then \( u(x,y) \) and \( v(x,y) \) are harmonic; that is, \( \Delta u = 0 \) and \( \Delta v = 0 \) for \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \).
   (b) Show that if \( u(x,y) \) is harmonic and \( C^2 \), then \( g(z) = \frac{\partial u}{\partial x}(x,y) - i\frac{\partial u}{\partial y}(x,y) \) is holomorphic.
   (c) Suppose \( G(z) \) is holomorphic and \( G'(z) = g(z) \). Show that there exists \( H(z) \) holomorphic with \( \text{Re}(H(z)) = u(x,y) \).

6. Rational functions
   (a) Let \( f(z) = p(z)/q(z) \) for \( p(z) \) and \( q(z) \) two polynomials in \( z \). Suppose that \( q(0) \neq 0 \). Show that \( f(z) \) has a power series expansion at zero \( f(z) = \sum_{k=0}^{\infty} c_k z^k \) with radius of convergence equal to the smallest norm among the roots of \( q(z) \).
   
   \text{Hint:} \ Use partial fractions. You may use (without proof) the Fundamental Theorem of Algebra and its corollary that every complex polynomial can be factored as \( c \prod (z - r_i) \) for (possibly repeated) roots \( r_i \in \mathbb{C} \) and some constant \( c \in \mathbb{C} \).
   (b) Give an example of a rational function of one real variable \( f(x) = p(x)/q(x) \) with \( q(0) \neq 0 \) such that the radius of convergence of its Taylor series is less than the smallest absolute value among the (real) roots of \( q(x) \).

7. Conformal maps
   (a) Suppose \( f(z) = u(x,y) + iv(x,y) \) is holomorphic, and consider the Jacobian matrix
   \[
   J_f = \begin{pmatrix}
   \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
   \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
   \end{pmatrix}
   \]
   Show that at a point \( z = x + iy \), there are real numbers \( \lambda \) and \( \phi \) such that \( J_f(x,y) \) is of the form
   \[
   J_f(z) = \lambda \begin{pmatrix}
   \cos \phi & -\sin \phi \\
   \sin \phi & \cos \phi
   \end{pmatrix}
   \]
   (b) Conclude that if \( J_f(x,y) \) is nonzero, it preserves angles. That is, given vectors \( v \) and \( w \), the angle between \( v \) and \( w \) is equal to the angle between \( J_f(x,y)v \) and \( J_f(x,y)w \).
   (c) Let \( U \subset \mathbb{C} \) be an open set. Suppose \( f: U \to \mathbb{C} \) has continuous partial derivatives in \( x \) and \( y \) and that \( J_f(x,y) \) is nonsingular and preserves angles for all \( z = x + iy \in U \) (such a map is called conformal). Show that \( f \) is holomorphic.