Inflection Points in Families of Algebraic Curves

Ashvin A. Swaminathan
Harvard College

A thesis submitted for the degree of
Artium Baccalaureus in Mathematics, with Honors
March 2017

Advised by
Joseph D. Harris and Anand P. Patel

Abstract

In this thesis, we discuss the theory of inflection points of algebraic curves from the perspective of enumerative geometry. The standard technique for counting inflection points — namely, computing Chern classes of the sheaves of principal parts — works quite well over smooth curves, but difficulties arise over singular curves. Our main results constitute a new approach to dealing with the problem of counting inflection points in one-parameter families of curves that have singular members. In the context of such families, we introduce a system of sheaves that serves to replace the sheaves of principal parts. The Chern classes of the new sheaves can be expressed as a main term plus error terms, with the main term arising from honest inflection points and the error terms arising from singular points. We explicitly compute the error term associated with a nodal singularity, and as a corollary, we deduce a lower bound on the error terms arising from other types of planar singularities. Our results can be used to answer a broad range of questions, from counting hyperflexes in a pencil of plane curves, to determining the analytic-local behavior of inflection points in a family of plane curves specializing to a singular curve, to computing the divisors of higher-order Weierstrass points in the moduli space of curves.

Acknowledgements

To begin with, I thank my advisor, Joe Harris, for suggesting the questions that led to this thesis and for providing me with the best guidance that I could have ever hoped for. I am indebted to him for his unparalleled patience and generosity in allowing me to meet with him every week for the past year and in offering me a glimpse of the wondrous world of algebraic geometry from his cultured perspective. Mathematics aside, Joe Harris is a paragon of humility and kindness; I would consider my life to be a great success if I could be but half the mathematician and human being that he is.

Next, I thank my co-advisor and collaborator Anand Patel for working shoulder-to-shoulder with me on the research component of this thesis. The "master of all things degree-3," Anand helped me to crystallize my raw ideas and served as a creative impetus at times when I did not know how to proceed. I am very grateful to him for his patience, flexibility, and optimism, without which my thesis endeavor would be nothing more than a dream.

My interest in mathematics was cemented in my freshman year, when I had the great privilege of taking Math 55 with Dennis Gaitsgory. To this day, I continue to be enamored with his technical facility, suave lecturing ability, and uncompromisingly rigorous approach to mathematics. I am grateful to Arul Shankar for teaching me all of the number theory that I know. I thank Noam Elkies for the many enjoyable conversations we have had about mathematics and music, for being my thesis reader, and for so graciously agreeing to collaborate with me on a fun little research paper last year. I also thank my math professors Tristan Collins, Mike Hopkins, Curtis McMullen, Alison Miller, and Wilfried Schmid, as well as my physics professors Jacob Barandes, Howard Georgi, Andrew Strominger, and Cumrun Vafa, for being some of the best teachers I have ever had.

James Tao deserves a special mention. His remarkable intuitive grasp of estoeric concepts never ceases to amaze me, and this thesis has benefited immensely from my conversations with him. Many thanks to Saahil Mehta for helping me discover a key improvement in the proof of one of the main results, and I am grateful to Saahil and Ben Lee for their friendship. It was very kind of Ravi Vakil to invite me to visit Stanford University and speak about this thesis in the Stanford Algebraic Geometry Seminar; his wonderful lecture notes taught me much of what I needed to know to write this thesis. Thanks also to Sabin Cautis, Aaron Landesman, Jacob Levinson, and Ziv Ran for helpful discussions on matters pertaining to this thesis.

I am grateful to Simon Rubinstein-Salzedo for introducing me to the world of mathematical research. Besides being fantastic research advisors, Joe Gallian and Ken Ono have been great sources of inspiration, advice, and help. I am grateful to both of them for providing me with the opportunity to hone my research skills at their respective REU programs. I thank Jesse Thorner and David Zureick-Brown for working closely with me during my two summers at the Emory REU.

Finally, I thank my amma, appa, pattis, and thathas for their unflinching faith in my abilities and for their undying love and support.

To amma and appa

Contents

Chapter 1

Introduction

> "The art of doing mathematics consists in finding that special case which contains all the germs of generality."
>
> David Hilbert, 1862–1943

1.1 Inflection Points: A Rudimentary Study

Given a function $f\colon \mathbb{R} \to \mathbb{R}$, when is the graph of f concave up or concave down? This sort of question is presented in a typical introductory calculus course as an example of how to use differentiation to determine the local behavior of a function. Specifically, if we take f to be twice-differentiable, we are taught that f is concave-up at a point $x \in \mathbb{R}$ if the second derivative $f''(x)$ is positive, because that means the first derivative f' is increasing at x; likewise, we are taught that f is concave-down at x if $f''(x) < 0$, because f' is decreasing. We conclude that the cases where $f''(x)$ is either strictly positive or negative provide us with useful geometric information about the graph of f. But what about the case where $f''(x) = 0$? It is not *a priori* clear how to interpret this case geometrically. As it happens, most basic calculus textbooks shy away from a complete discussion of this issue by restricting their consideration to points $x \in \mathbb{R}$ at which f'' not only equals 0, but also changes sign, so that the graph of f switches between being concave-up and concave-down at x (colloquially speaking, points at which "the foot releases the gas pedal and applies the brake"). We are taught that the corresponding points

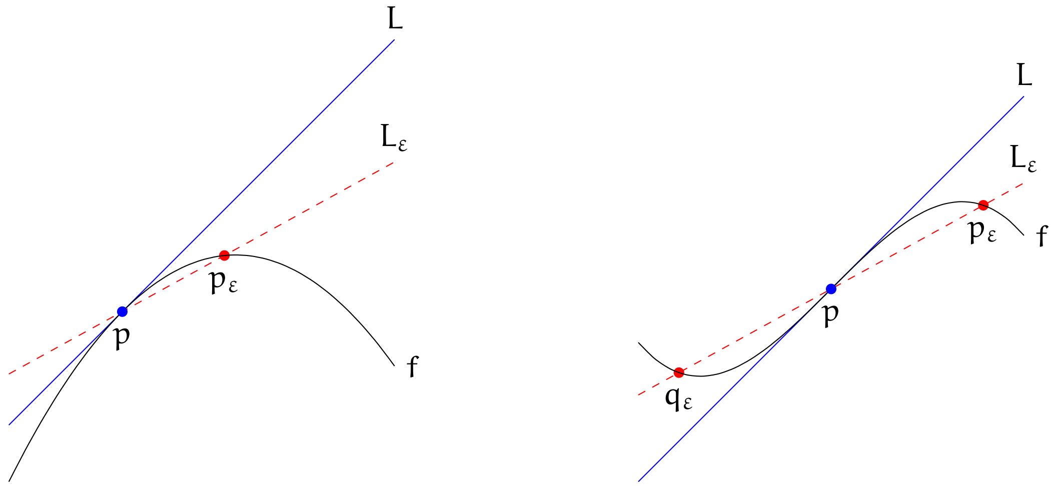

Figure 1.1: Both diagrams depict the limiting behavior of the secant line L_ε (in red) as it tends toward the tangent line L (in blue) to the graph of f (in black) at the point $p = (x, f(x))$. However, in diagram (a), the point p is not in inflection point, whereas in diagram (b), the point p is an inflection point.

$p = (x, f(x))$ on the graph of f are called **inflection points**, but we learn little beyond this basic definition. We almost certainly hear nothing of the fact that there is a comprehensive theory that not only vastly generalizes this rudimentary notion of inflection point, but also allows us to draw many interesting conclusions about the behavior of such points. Suffice it to say that "inflection point" is just another one of those terms that we must memorize in order to deem ourselves as competent in basic calculus.

Nevertheless, there is much more to being an inflection point than the above definition would suggest. As depicted in Figure 1.1, consider a function $f\colon \mathbb{R} \to \mathbb{R}$ and a point $x \in \mathbb{R}$ such that $p = (x, f(x))$ is *not* an inflection point of the graph of f. For each $0 < \varepsilon \ll 1$, draw the secant line L_ε joining the pair of points p and $p_\varepsilon = (x + \varepsilon, f(x + \varepsilon))$. In the limit as $\varepsilon \to 0$, the secant lines L_ε converge, essentially by definition, toward the tangent line L to the graph of f at p; furthermore, the two points p and p_ε can be said to "come together" in this limit. Now that may seem to be a non-rigorous statement, and it certainly is, but as we illustrate in § 1.2.1, there is a way of making precise sense of the observation that two distinct points are coming together: we say that the multiplicity of the intersection between L and the graph of f at p is 2. On the other hand, suppose that p *is* an inflection point. Notice that each of the lines L_ε now meets the graph of f in a third point q_ε that lies "just to the left" of the point p on the graph of f, and in the limit as $\varepsilon \to 0$, all three of the points p, p_ε, and q_ε come together as the secant lines L_ε

approach the tangent line L. We say that the multiplicity of the intersection between L and the graph of f at p is 3, because this intersection arises as the limit of three points coming together.

Given the observations made in the preceding paragraph, it is natural to ask whether the condition that L meets the graph of f with multiplicity greater than 2 at p is not merely a necessary, but also a sufficient condition for p to be an inflection point. The answer to this question is no, but only for a superficial reason: the condition that L meets the graph of f with multiplicity greater than 2 at p does imply that $f''(x) = 0$, but does not imply that f'' changes sign at x. Rather than introduce a new term to describe points at which the second derivative vanishes but does not change sign,[1] we shall abuse terminology by redefining the term "inflection point" to mean all points at which f'' vanishes.

Our revised definition of inflection point is the notion that is generally adopted in the field of **algebraic geometry**. In this thesis, we shall approach the study of inflection points exclusively from the perspective of algebraic geometry, so that instead of considering the graphs of functions $f\colon \mathbb{R} \to \mathbb{R}$, we shall be working with algebraic curves. The advantages of working in the realm of algebraic geometry are numerous; perhaps the most significant benefit is that we can appeal to sophisticated machinery to turn difficult geometric problems, like describing inflection points on curves, into tractable algebraic ones, like computing dimensions of vector spaces.

We conclude this prefatory section with the following question, which is the first of many "Motivating Questions" that appear throughout this thesis as a means of providing the reader with a general sense of direction on our journey to acquire a deep and thorough understanding of inflection points of curves.

Motivating Question 1. How do we define inflection points on algebraic curves? Given a curve, can we identify its inflection points, and if not, in what ways can we characterize them?

1.2 A More Sophisticated Approach

The first step in our investigation of Motivating Question 1 is to make precise the notion of intersection multiplicity, which lies at the crux of our revised definition of in-

[1] Such points are sometimes called **undulation points**.

flection point. We then use the precise definition of intersection multiplicity to define inflection points for algebraic curves in the plane. We conclude this section by using an important classical construction, called the Hessian, to obtain a complete description of the inflection points of such curves. Before we proceed, we establish the following basic conventions, which hold unless otherwise specified.

- k denotes an algebraically closed field of characteristic 0.
- By "curve," we mean 1-dimensional variety over k, and by "plane curve" we mean 1-dimensional subvariety of the projective plane $\mathbb{P}^2_k$.
- All schemes are k-schemes, and all points are k-valued.

1.2.1 What is Intersection Multiplicity?

Let $C \subset \mathbb{P}^2_k$ be a plane curve with the property that no irreducible component of C is a line.[2] By viewing the tangent line L to a smooth point $p \in C$ as the limit of secant lines, we reasoned in § 1.1 that L meets C at p with multiplicity at least 2, and we further claimed that this multiplicity is greater than 2 if and only if p is an inflection point of C. The language of schemes provides us with just the tools that we need to make these statements precise.

In defining the intersection multiplicity of C and L at p, it makes sense to look at their scheme-theoretic intersection, denoted $C \cap L$. Since $C \cap L$ is necessarily 0-dimensional, it comes equipped with a natural notion of multiplicity at a point, namely the dimension of the local ring as a k-vector space. Guided by this intuition, we define intersection multiplicity as follows.

Definition 2. With C as above, let $p \in C$ be a smooth point, and let $L \subset \mathbb{P}^2_k$ be a line meeting C at p. The **intersection multiplicity** $\mathrm{mult}_p(C, L)$ of C and L at p is simply the multiplicity of $C \cap L$ at p, and is given explicitly by

$$\mathrm{mult}_p(C, L) = \dim_k \mathcal{O}_{C \cap L, p},$$

where $\dim_k$ denotes dimension as a k-vector space.[3]

[2] This condition is reasonable because lines cannot have inflection points; also, it will become apparent that our analysis of inflection points on C breaks down if we allow C to have a linear component.

[3] Note that we are abusing notation here by using the symbol p to denote what is set-theoretically speaking "the same" point of $C \cap L$, C, L, and $\mathbb{P}^2_k$.

Remark 3. If Definition 2 seems like it has come out of nowhere, the reader might be relieved to know that finding a good notion of intersection multiplicity was historically a challenging problem: indeed, it occupied algebraic geometers for the first half of the twentieth century (see [EH16, § 1.3.8] for an in-depth discussion about multiplicity).

To check that Definition 2 is reasonable, we must use it to verify the basic geometric intuitions described in § 1.1. The following lemma confirms our assertion that a curve has intersection multiplicity at least 2 with every one of its tangent lines.

Lemma 4. *With notation as in Definition 2, we have* $\mathrm{mult}_p(C, L) \geq 1$. *Furthermore, we have* $\mathrm{mult}_p(C, L) \geq 2$ *if and only if* L *is the tangent line to* C *at* p.

Proof. Since the statement of the lemma is purely local with respect to the point p, it suffices to work in a standard affine patch $\operatorname{Spec} k[x, y] = \mathbb{A}^2_k \subset \mathbb{P}^2_k$ containing p. Suppose on this patch that C is given by the vanishing locus of a polynomial $f(x, y) \in k[x, y]$, that L is given by the vanishing locus of a linear polynomial $ax + by + c$ for some $a, b, c \in k$ satisfying (without loss of generality) $b \neq 0$, and that p is expressed in these coordinates as $p = (p_1, p_2)$. Then $p_2 = -\frac{a}{b}p_1 - \frac{c}{b}$, and so p_1 is a root of the polynomial $g(x) = f\big(x, -\frac{a}{b}x - \frac{c}{b}\big) \in k[x]$. The scheme-theoretic intersection $C \cap L$ is simply given by the fiber product $C \cap L = C \times_{\mathbb{P}^2_k} L$, so it follows that

$$\begin{aligned}\mathcal{O}_{C\cap L}(\mathbb{A}^2_k) &= k[x, y]/(f(x, y)) \otimes_{k[x,y]} k[x, y]/(ax + by) \\ &= k[x]/(g(x)).\end{aligned}$$

We deduce that the local ring of $C \cap L$ at p is given by

$$\mathcal{O}_{C\cap L,p} = k[x]_{(x-p_1)}/(g(x)).$$

It is now a simple matter to compute the intersection multiplicity of C and L at p. Indeed, we have that

$$\mathrm{mult}_p(C, L) = \dim_k \mathcal{O}_{C\cap L,p} = \dim_k k[x]_{(x-p_1)}/(g(x)),$$

and this last quantity is readily seen to be the order of vanishing of $g(x)$ at $x = p_1$, by which we mean the number of times that $x - p_1$ occurs as a factor in the (unique) factorization of $g(x)$ into linear polynomials over k. Since p_1 is a root of $g(x)$, we see that $\mathrm{mult}_p(C, L) \geq 1$. Now, the order of vanishing of $g(x)$ at $x = p_1$ is at least 2 if and only if $x = p_1$ is also a root of the derivative $g'(x)$. But notice that

$$\begin{aligned} g'(x) &= \frac{d}{dx} f\left(x, -\tfrac{a}{b}x - \tfrac{c}{b}\right) \\ &= \left.\frac{\partial f}{\partial x}\right|_{\left(x, -\frac{a}{b}x - \frac{c}{b}\right)} + \left.\frac{\partial f}{\partial y}\right|_{\left(x, -\frac{a}{b}x - \frac{c}{b}\right)} \cdot -\frac{a}{b}. \end{aligned} \tag{1.1}$$

Moreover, by definition, L is tangent to C at p if and only if

$$\left.\frac{\partial f}{\partial x}\right|_{(p_1, p_2)} \bigg/ \left.\frac{\partial f}{\partial y}\right|_{(p_1, p_2)} = \frac{a}{b}. \tag{1.2}$$

Then, by combining (1.1) and (1.2), we find that $g'(p_1) = 0$, or equivalently that $\text{mult}_p(C, L) \geq 2$, if and only if L is tangent to C at p. □

Remark 5. In the course of proving Lemma 4, we secretly confirmed another claim made in § 1.1. Take $x \in \mathbb{R}$, and suppose that f has a Taylor series expansion at x that converges on some small neighborhood of x. Then for all t in this neighborhood, we can write

$$y = f(t) = f(x) + f'(x) \cdot (t - x) + \frac{f''(x)}{2!} \cdot (t - x)^2 + \frac{f'''(x)}{3!} \cdot (t - x)^3 + \cdots .$$

Observe that the equation of the tangent line to the graph of f at x is given by

$$y = f'(x) \cdot (t - x) - f(x),$$

so substituting this expression for y into the series expansion of f yields

$$0 = \frac{f''(x)}{2!} \cdot (t - x)^2 + \frac{f'''(x)}{3!} \cdot (t - x)^3 + \cdots .$$

It is now clear that $f''(x) = 0$ if and only if the right-hand side above vanishes to order at least 3 at $t = x$. More generally, for any $m \geq 2$, we find that i^{th} derivative of f at x is 0 for all $i \in \{2, \dots, m\}$ if and only if the right-hand side above vanishes to order at least $m + 1$ at $t = x$. But recall that in the proof of the lemma, we showed that when the curve C vanishes to order $m + 1$ along the line L at the point p, then the intersection multiplicity of C and L at p is equal to $m + 1$. It follows that the condition $f''(x) = 0$ is equivalent to the condition that the intersection multiplicity of the graph and its tangent line is at least 3, as we had previously asserted.

We are now in position to define inflection points of plane curves. As the following definition suggests, the word "flex" is commonly used in place of "inflection" in the context of plane curves.

Definition 6. Let $C \subset \mathbb{P}^2_k$ be a plane curve, no irreducible component of which is a line. Let $p \in C$ be a smooth point, and let L be the line tangent to C at p. We say that p is a **flex** if $\text{mult}_p(C, L) \geq 3$, that p is a **hyperflex** if $\text{mult}_p(C, L) \geq 4$, and more generally that p is an m^{th}**-order flex** if $\text{mult}_p(C, L) \geq m$ for some integer $m \geq 3$.

1.2.2 Case Study: Flexes on a Plane Curve

The characterization of flexes provided in Definition 6 has the advantage that it lends itself more easily to geometric interpretation. However, this definition lacks practical value — although it tells us how to check whether any given point on a curve is a flex, it fails to describe how we might go about finding all of the flexes on a given curve. The purpose of this section is to present the method that classical algebraic geometers first developed in their quest to study flexes on plane curves; one of the earliest published records of the argument that follows can be found in an 1879 treatise written by G. Salmon (see [Sal60]).

Let $C \subset \mathbb{P}^2_k$ be a plane curve, no irreducible component of which is a line. Taking homogeneous coordinates $[X_1 : X_2 : X_3]$ on $\mathbb{P}^2_k$ and observing that C is by definition a hypersurface in $\mathbb{P}^2_k$, we have that C can be expressed as the vanishing locus of a homogeneous polynomial $F(X_1, X_2, X_3) \in k[X_1, X_2, X_3]$. We can then associate to C its **Hessian** $H(C)$, which is the closed subscheme of $\mathbb{P}^2_k$ defined by the vanishing of the determinant of the 3×3 symmetric matrix of second-order partial derivatives of $F(X_1, X_2, X_3)$. Explicitly, $H(C)$ is equal to the vanishing locus of the polynomial

$$H_F(X_1, X_2, X_3) = \begin{vmatrix} \frac{\partial^2 F}{\partial X_1^2} & \frac{\partial^2 F}{\partial X_1 \partial X_2} & \frac{\partial^2 F}{\partial X_1 \partial X_3} \\ \frac{\partial^2 F}{\partial X_1 \partial X_2} & \frac{\partial^2 F}{\partial X_2^2} & \frac{\partial^2 F}{\partial X_2 \partial X_3} \\ \frac{\partial^2 F}{\partial X_1 \partial X_3} & \frac{\partial^2 F}{\partial X_2 \partial X_3} & \frac{\partial^2 F}{\partial X_3^2} \end{vmatrix} \in k[X_1, X_2, X_3]. \tag{1.3}$$

As with any construction that seems to depend on a choice of coordinates, we ought to show that the Hessian is in fact independent of this choice.[4] The next lemma es-

[4]If anything, we ought to justify our notation, which presupposes that $H(C)$ does not depend on the choice of coordinates!

tablishes that the Hessian is invariant under the action of the projective linear group $\mathrm{PGL}_3(k)$, the group of automorphisms of $\mathbb{P}^2_k$.

Lemma 7. *Let* $F(X_1, X_2, X_3) \in k[X_1, X_2, X_3]$, *and let* $M \in \mathrm{PGL}_3(k)$. *Then for every point* $[A_1 : A_2 : A_3] \in \mathbb{P}^2_k$, *we have* $H_F(A_1, A_2, A_3) = 0$ *if and only if* $H_{F \circ M^{-1}}(M(A_1, A_2, A_3)) = 0$.

Proof. Take any point $[A_1 : A_2 : A_3] \in \mathbb{P}^2_k$. By abuse of notation, let M denote a representative in $\mathrm{GL}_3(k)$ of the given transformation $M \in \mathrm{PGL}_3(k)$, and let the row-i, column-j entry of M^{-1} be denoted by $M_{i,j}$. An easy way to evaluate the desired quantity $H_{F \circ M^{-1}}(M([A_1 : A_2 : A_3]))$ is to apply Faà di Bruno's formula for higher-order derivatives of multivariable functions. In our situation, the formula tells us that

$$\left.\frac{\partial^2 (F \circ M^{-1})}{\partial X_i \partial X_j}\right|_{M(A_1, A_2, A_3)} = \sum_{k,\ell=1}^{3} \left.\frac{\partial^2 F}{\partial X_i \partial X_j}\right|_{(A_1, A_2, A_3)} M_{k,i} M_{\ell,j},$$

from which it is evident that

$$H_{F \circ M^{-1}}(M([A_1 : A_2 : A_3])) = (\det M^{-1})^2 \cdot H_F(A_1, A_2, A_3).$$

Since $\det M \neq 0$, the lemma follows. □

It turns out to be useful to know what the Hessian looks like in a standard affine patch of $\mathbb{P}^2_k$; the next lemma provides us with this information.

Lemma 8. *Let* $C \subset \mathbb{P}^2_k$ *be a plane curve of degree* d, *given by the vanishing locus of a homogeneous polynomial* $F(X_1, X_2, X_3) \in k[X_1, X_2, X_3]$, *and let* $f(x_1, x_2) \in k[x_1, x_2]$ *be the polynomial defined by* $f(x_1, x_2) = F(x_1, x_2, 1)$. *In the standard affine patch* $\operatorname{Spec} k[x_1, x_2] = \mathbb{A}^2_k \subset \mathbb{P}^2_k$ *where* $X_3 \neq 0$, *the Hessian* $H(C)$ *is given by the vanishing locus of the polynomial*

$$h_f(x_1, x_2) = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & (d-1)\frac{\partial f}{\partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & (d-1)\frac{\partial f}{\partial x_2} \\ (d-1)\frac{\partial f}{\partial x_1} & (d-1)\frac{\partial f}{\partial X_2} & d(d-1)f \end{vmatrix} \in k[x_1, x_2].$$

Proof. To obtain the desired expression, we perform one column operation and one row operation, which do not alter the determinant, to the matrix in (1.3). First, observe that by adding X_1 times the first column plus X_2 times the second column to X_3 times the third column, we have by Euler's Lemma for homogeneous functions that

$$X_3 \cdot H_F(X_1, X_2, X_3) = \begin{vmatrix} \frac{\partial^2 F}{\partial X_1^2} & \frac{\partial^2 F}{\partial X_1 \partial X_2} & (d-1)\frac{\partial F}{\partial X_1} \\ \frac{\partial^2 F}{\partial X_1 \partial X_2} & \frac{\partial^2 F}{\partial X_2^2} & (d-1)\frac{\partial F}{\partial X_2} \\ \frac{\partial^2 F}{\partial X_1 \partial X_3} & \frac{\partial^2 F}{\partial X_2 \partial X_3} & (d-1)\frac{\partial F}{\partial X_3} \end{vmatrix}.$$

Second, observe that by adding X_1 times the first row plus X_2 times the second row to X_3 times the third row, we have by Euler's Lemma that

$$X_3^2 \cdot H_F(X_1, X_2, X_3) = \begin{vmatrix} \frac{\partial^2 F}{\partial X_1^2} & \frac{\partial^2 F}{\partial X_1 \partial X_2} & (d-1)\frac{\partial F}{\partial X_1} \\ \frac{\partial^2 F}{\partial X_1 \partial X_2} & \frac{\partial^2 F}{\partial X_2^2} & (d-1)\frac{\partial F}{\partial X_2} \\ (d-1)\frac{\partial F}{\partial X_1} & (d-1)\frac{\partial F}{\partial X_2} & d(d-1)F \end{vmatrix}.$$

Then, setting $(X_1, X_2, X_3) = (x_1, x_2, 1)$ yields the desired expression. □

Now that we have proven some basic facts about the Hessian, it is natural to wonder why we bothered to introduce it in the first place: what does all of this work buy us? As we show in the next proposition, the Hessian $H(C)$ has the seemingly magical property that its intersection with the curve C is precisely the locus of flexes of C.

Proposition 9. *Let $C \subset \mathbb{P}_k^2$ be a plane curve, no irreducible component of which is a line. Then a smooth point p is a flex of C if and only if p also lies on $H(C)$. Moreover, if p is an m^{th}-order flex of C but not an $(m+1)^{\text{st}}$-order flex, then the intersection multiplicity of C and $H(C)$ at p is equal to $m-2$.*

Proof. Let d be the degree of C, and let p be a smooth point on C. Lemma 7 allows us to reconfigure our geometric setup via projective linear transformations, so in particular, we may assume that p has homogeneous coordinates $p = [0 : 0 : 1]$ and that the tangent line to C at p is the vanishing locus of X_2. It follows that in the standard affine patch $\operatorname{Spec} k[x_1, x_2] = \mathbb{A}_k^2 \subset \mathbb{P}_k^2$ where $X_3 \neq 0$, the curve C is the vanishing locus of a polynomial $f(x_1, x_2) \in k[x_1, x_2]$ that may be written as

$$f(x_1, x_2) = a_{01}x_2 + \sum_{i+j \geq 2} a_{ij} x_1^i x_2^j,$$

where $a_{ij} \in k$ for all i, j and $a_{01} \neq 0$.

We begin with the first part of the proposition. The condition that p is a flex of C is equivalent to stipulating that the polynomial $f(x_1, 0) \in k[x_1]$ vanishes to order at least 3 at $x_1 = 0$, which is equivalent to the condition that $a_{20} = 0$. But by Lemma 8, we have

$$h_f(0,0) = \begin{vmatrix} 2a_{20} & a_{11} & (d-1)a_{10} \\ a_{11} & 2a_{02} & (d-1)a_{01} \\ (d-1)a_{10} & (d-1)a_{01} & d(d-1)a_{00} \end{vmatrix} = -2a_{01}^2 a_{20}(d-1)^2,$$

which vanishes precisely when $a_{20} = 0$ because $a_{01} \neq 0$. We conclude that p is a flex of C if and only if p also lies on $H(C)$.

We now handle the second part of the proposition. The condition that p is an m^{th}-order flex of C but not an $(m+1)^{\text{st}}$-order flex is equivalent to stipulating that the polynomial $f(x_1, 0) \in k[x_1]$ vanishes to order exactly m at $x_1 = 0$, which is equivalent to the condition that $a_{i0} = 0$ for all $i \in \{2, \dots, m-1\}$ and $a_{m0} \neq 0$. As in Lemma 8, let $h_F(x_1, x_2) \in k[x_1, x_2]$ be the polynomial cutting out the Hessian in the chosen patch. Then the intersection multiplicity of the curve and its Hessian at p is given by

$$\text{mult}_p(C, H(C)) = \dim_k \mathcal{O}_{C \cap H(C), p} = \dim_k k[x_1, x_2]_{(x_1, x_2)} / (f(x_1, x_2), h_F(x_1, x_2)).$$

In the local ring $k[x_1, x_2]_{(x_1, x_2)}$, the equality $f(x_1, x_2) = 0$ may be rewritten as

$$x_2 = \left(\sum_{\substack{i+j \geq 2 \\ j \geq 1}} a_{ij} x_1^i x_2^{j-1} \right)^{-1} \cdot \left(\sum_{i \geq 2} a_{i0} x_1^i \right),$$

where we have used the fact that $a_{01} \neq 0$ to invert the sum on the right-hand side of the above. Thus, the condition that $a_{i0} = 0$ for all $i \in \{2, \dots, m-1\}$ implies that $x_2 = \alpha \cdot x_1^m$, where α is a unit in the local ring $k[x_1, x_2]_{(x_1, x_2)}$. We therefore have that

$$\text{mult}_p(C, H(C)) = \dim_k k[x_1]_{(x_1)} / (h_F(x_1, x_1^m)).$$

The proposition statement suggests that the above dimension should be $m - 2$, so to check whether this is indeed the case, we might as well simplify our work by quotienting out by the relation $x_1^m = 0$. More precisely, to prove the proposition, it suffices to show that the lowest-degree term of $h_f(x_1, 0)$ is proportional to x_1^{m-2}. Appealing to

Lemma 8, we obtain the following equality:

$$h_f(x_1,0) = \begin{vmatrix} \sum_{i\geq m} i(i-1)a_{i0}x_1^{i-2} & \sum_{i\geq 2}(i-1)a_{(i-1)1}x_1^{i-2} & \sum_{i\geq m} i(d-1)a_{i0}x_1^{i-1} \\ \sum_{i\geq 2}(i-1)a_{(i-1)1}x_1^{i-2} & \sum_{i\geq 2} 2a_{(i-2)2}x_1^{i-2} & \sum_{i\geq 1}(d-1)a_{(i-1)1}x_1^{i-1} \\ \sum_{i\geq m} i(d-1)a_{i0}x_1^{i-1} & \sum_{i\geq 1}(d-1)a_{(i-1)1}x_1^{i-1} & \sum_{i\geq m} d(d-1)a_{i0}x_1^{i} \end{vmatrix}.$$

The above determinant may appear to be rather unwieldy, but it is evident that its expansion contains no terms of degree less than $m-2$ in x_1. Thus, all we care about is isolating the x_1^{m-2} term, and by eliminating terms of higher degree in x_1, it is easy to verify that the coefficient of this term is given by

$$-m(m-1)a_{m0}a_{01}^2(d-1)^2,$$

which is nonzero because a_{m0} and a_{01} were assumed to be nonzero. Thus, we have that $\text{mult}_p(C, H(C)) = m-2$, as desired. □

The result of Proposition 9 may seem miraculous: we pulled the Hessian out of what seems like nowhere, and it somehow managed to pick out the flexes on a plane curve, with such precision that the intersection multiplicity of the curve and its Hessian at a flex determines the order of that flex. Indeed, the Hessian is a mysterious *ad hoc* construction that does not seem to generalize in any natural way to studying inflection points on objects other than plane curves. It is thus reasonable to ask the following basic question.

Motivating Question 10. Can we find a more direct, intuitive, and general strategy for studying inflection points?

In our attempt to answer Motivating Question 10, we will be forced to redefine what we mean by "study": in more general situations than describing flexes of plane curves, we cannot hope to obtain such an explicit description of the locus of inflection points. Fortunately for us, **intersection theorists** (the particular breed of algebraic geometer that studies inflection points and the like) are not so much interested in determining which points on a curve are flexes as they are in counting how many flexes a curve has.[5] We therefore arrive at the following question.

[5]Historical aside: One of the first **intersection theorists** was C. Maclaurin, who worked on Bézout's Theorem (see Theorem 33) about a century after R. Descartes invented the coordinate system in 1637; for more background about intersection theory, refer to [Kle85].

Motivating Question 11. Given a geometric object that exhibits inflectionary behavior, can we say anything about how many inflection points it has?[6]

As it happens, we can actually use our results about the Hessian to answer Motivating Question 11 in the context of plane curves. To see how, observe that the Hessian $H(C)$ of a plane curve C of degree d either vanishes on all of $\mathbb{P}^2_k$ or is a (not necessarily reduced) curve of degree $3(d-2)$. In the latter case, Bézout's Theorem (see Theorem 33) implies that the total intersection multiplicity between C and $H(C)$ is $3d(d-2)$, as long as C does not share a component with $H(C)$. If we further assume that C has no hyperflexes, Proposition 9 tells us that $C \cap H(C)$ has multiplicity equal to 1 at each flex, implying that C has exactly $3d(d-2)$ flexes! This would be a fascinating result, but we need to check whether the conditions that it relies upon are mild enough to make the result useful.

The conditions that $H(C)$ does not vanish everywhere and that C does not share a component with $H(C)$ turn out to be irrelevant. In both of these cases, notice that C must have infinitely many flexes. We know that this is possible when C contains a line, because every point of a line is a flex (in the sense that the tangent line vanishes to all orders at the point), but can it happen when C has no linear component? The following proposition tells us that the answer is, fortunately, no.

Proposition 12. *Let $C \subset \mathbb{P}^2_k$ be a plane curve, no irreducible component of which is a line. Then C has finitely many flexes. Equivalently, no component of C is contained in $H(C)$.*

Proof. Suppose C has infinitely many flexes. Then by Proposition 9, C meets $H(C)$ at infinitely many points, so there is an irreducible component $D \subset C$ that is contained in $H(C)$. Furthermore, D must be smooth, and in particular reduced, because we stipulated that flex points must be smooth in Definition 6. Moreover, since no irreducible component of C is a line, there must be infinitely many flexes on the open subscheme of C where $X_3 \neq 0$ (where we have taken homogeneous coordinates $[X_1 : X_2 : X_3]$ on $\mathbb{P}^2_k$), so we may reduce to the situation where all of the curves under consideration lie in the standard affine patch $\operatorname{Spec} k[x_1, x_2] = \mathbb{A}^2_k \subset \mathbb{P}^2_k$.

Let $p \in D$ be a point. Appealing to Lemma 7, we may assume that p is given by $x_1 = x_2 = 0$. Since p is a smooth point, the completion $\widehat{\mathcal{O}}_{D,p}$ of the local ring of D at p is isomorphic to $k[[t]]$, the power series ring in one variable over the residue field of

[6]The subfield of algebraic geometry concerned with counting geometric objects with certain properties is called **enumerative geometry**.

D at p. Regardless of how we choose the variable t, it must be the case that under the associated embedding $\mathcal{O}_{D,p} \hookrightarrow \widehat{\mathcal{O}}_{D,p}$, our affine coordinates are given by $x_1 = t^{e_1}\varepsilon_1(t)$ and $x_2 = t^{e_2}\varepsilon_2(t)$, where e_1, e_2 are positive integers with $\min\{e_1, e_2\} = 1$ and $\varepsilon_1(t), \varepsilon_2(t)$ are elements of $k[[t]]^\times$. We may assume without loss of generality that $e_1 = 1$, which implies that we may take x_1 to be a uniformizer for $\widehat{\mathcal{O}}_{D,p}$. Thus, in the completed local ring $\widehat{\mathcal{O}}_{D,p}$, we can express x_2 as a power series in x_1 with the property that if $f(x_1, x_2) \in k[x_1, x_2]$ is the polynomial whose vanishing locus is D, then $f(x_1, x_2(x_1)) = 0$. Differentiating with respect to x_1, we find that

$$\frac{dx_2}{dx_1} = -\frac{\partial f}{\partial x_1}\bigg/\frac{\partial f}{\partial x_2},$$

where we are evaluating all derivatives at the point $(x_1, x_2(x_1))$. There is no need to worry in the above equation as to whether $\frac{\partial f}{\partial x_2} = 0$ when evaluated at $(x_1, x_2(x_1))$. Indeed, this cannot happen because otherwise, we would have $\frac{\partial f}{\partial x_1} = 0$ when evaluated at $(x_1, x_2(x_1))$ as well. Then, the injectivity of the map $\mathcal{O}_{D,p} \hookrightarrow \widehat{\mathcal{O}}_{D,p}$ implies that $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0$ at the point p, which would contradict the fact that p is a smooth point.

Differentiating with respect to x_1 for the second time, we find that

$$\begin{aligned}\frac{d^2x_2}{dx_1^2} &= \left(\frac{\partial f}{\partial x_1}\frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_1^2} + \left(\frac{\partial f}{\partial x_1}\frac{\partial^2 f}{\partial x_2^2} - \frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)\frac{dx_2}{dx_1}\right)\bigg/\left(\frac{\partial f}{\partial x_2}\right)^3 \\ &= \left(\frac{1}{(d-1)^2}h_f - \frac{d}{d-1}f\left(\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2\right)\right)\bigg/\left(\frac{\partial f}{\partial x_2}\right)^3,\end{aligned}$$

where in the last step above, we applied Lemma 8 (note that we are still evaluating the expression above at the point $(x_1, x_2(x_1))$). Since $h_f(x_1, x_2) = f(x_1, x_2) = 0$ as elements of $\mathcal{O}_{D,p}$, we must also have $h_f(x_1, x_2(x_1)) = f(x_1, x_2(x_1)) = 0$ as elements of $\widehat{\mathcal{O}}_{D,p}$. Substituting these identities into the above equation yields that

$$\frac{d^2x_2}{dx_1^2} = 0,$$

where the above is an equality of power series. Therefore, all derivatives of order 2 or higher of x_2 with respect to x_1 are 0, so since p is given by $x_1 = x_2 = 0$, we have that $x_2 = a \cdot x_1$ as elements of $\widehat{\mathcal{O}}_{D,p}$ for some $a \in k$. Again appealing to the injectivity of the map $\mathcal{O}_{D,p} \hookrightarrow \widehat{\mathcal{O}}_{D,p}$ implies that $x_2 = a \cdot x_1$ as elements of $\mathcal{O}_{D,p}$. But then the condition that $f(x_1, x_2(x_1)) = 0$ implies that $(x_2 - a \cdot x_1) \mid f(x_1, x_2)$, so because D is irreducible,

$f(x_1, x_2) = b \cdot (x_2 - a \cdot x_1)$ for some $b \in k$. It follows that D is a line, as desired. □

Remark 13. The key idea that we employed in the proof of Proposition 12 is to express the affine coordinates x_1 and x_2 as power series in a variable t that uniformizes the completed local ring $\widehat{\mathcal{O}}_{D,p}$; we then observed that x_1 was also a uniformizer for $\widehat{\mathcal{O}}_{D,p}$, which allowed us to express x_2 as a power series in x_1. This algebraic construction has a very natural geometric interpretation: one can view the variable x_1 as a sort of "local parameter" on the curve, and one can further view the curve D as being given locally by the graph of x_2 as an analytic function in the parameter x_1. This technique of working in an **analytic-local neighborhood** turns out to be central to our consideration of inflection points.

The condition that C has no hyperflexes is more problematic, as it will force us to throw out some curves. However, as the next proposition illustrates, most curves do not have hyperflexes, so we shall not be excluding too many curves!

Proposition 14. *A general plane curve of degree* $d > 1$ *has no hyperflexes.*[7]

Proof. The result is obvious for curves of degree $d = 2, 3$ (there are no lines meeting such curves with multiplicity at least 4), so we may assume that $d \geq 4$. Recall that the parameter space of all plane curves of degree d is a projective space of dimension $N(d) := \binom{d+2}{2} - 1$; we denote this parameter space by $\mathbb{P}_k^{N(d)}$. Now, let

$$Y = \mathbb{P}_k^2 \times \mathbb{P}_k^{N(1)} \times \mathbb{P}_k^{N(d)}$$

be the space of triples of points, lines, and degree-d curves in $\mathbb{P}_k^2$, and let

$$X = \{(\text{point } p, \text{line } L, \text{curve } C) \in Y : \ \text{mult}_p(L, C) \geq 4\}$$

be the space of triples of hyperflex points along with their tangent lines and the curves that contain them.[8] It is not hard to see that X is a closed subvariety of Y (to check this, one simply writes down the equations that cut out X). Similarly, one can check that

$$Z = \{(\text{point } p, \text{line } L) \in \mathbb{P}_k^2 \times \mathbb{P}_k^{N(1)} : p \in L\}$$

[7] Recall that a **general** element of a scheme has a property if that property holds on every point of a dense open subscheme.

[8] The locus X is sometimes called an **incidence correspondence**.

is a closed subvariety of $\mathbb{P}^2_k \times \mathbb{P}^{N(1)}_k$. Moreover, by [Har95, Corollary 11.13 and Theorem 11.14], we have that Z is irreducible of dimension 3, because all of the fibers of the projection map $Z \to \mathbb{P}^2_k$ are copies of $\mathbb{P}^1_k$. Now, notice that the projection map $X \to \mathbb{P}^2_k \times \mathbb{P}^{N(1)}_k$ has image equal to Z. Applying [Har95, Corollary 11.13 and Theorem 11.14] once again to the resulting map $X \to Z$, we have that X is irreducible of dimension $N(d) - 1$, because all of the fibers of the map $X \to Z$ are copies of $\mathbb{P}^{N(d)-4}_k$ (this relies on the assumption $d \geq 4$). But then, the image of X under the projection map $X \to \mathbb{P}^{N(d)}_k$ has codimension at least 1, so there is a (necessarily dense) open subscheme of $\mathbb{P}^{N(d)}_k$ such that the corresponding curves have no hyperflexes. □

Therefore, when $d > 1$, a dense open subscheme of the parameter space of plane curves of degree d has the property that the corresponding curves contain no hyperflexes. Moreover, the curves that do not contain lines also form a dense open, and the intersection of these two dense opens is itself a dense open. Combining all of the results heretofore proven, we arrive at the following elegant fact.

Corollary 15. *A general plane curve of degree $d > 1$ has exactly $3d(d-2)$ flexes and has no hyperflexes.*

The following example serves to explain what Corollary 15 means in the two simplest cases, where $d = 2, 3$.

Example 16. When $d = 2$, Corollary 15 tells us that a general plane conic curve has zero flexes. In fact, no plane conic curve has a flex — it is not possible for a line to meet a conic with multiplicity greater than 2 at a point.

When $d = 3$, Corollary 15 tells us that a general plane cubic curve has nine flexes. An **elliptic curve** $E \subset \mathbb{P}^2_k$ is a smooth plane curve of degree 3, with the choice of a point that serves as the identity element for a group structure on the set of points of E; recall that this group structure is characterized by the property that $a + b + c = 0$ for points $a, b, c \in E$ if and only if there exists a line cutting out the divisor $a + b + c$ on E. It turns out that the 3-torsion subgroup of E, namely the subgroup of points $a \in E$ such that $a + a + a = 0$ under the group law, is always isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$; in particular, E has exactly nine 3-torsion points. But $a \in E$ is a three-torsion point if and only if there is a line having intersection multiplicity equal to 3 with E at a, so the 3-torsion points of E are precisely its flexes. It is fascinating that we arrive at the same number of nine flexes from two seemingly disparate perspectives!

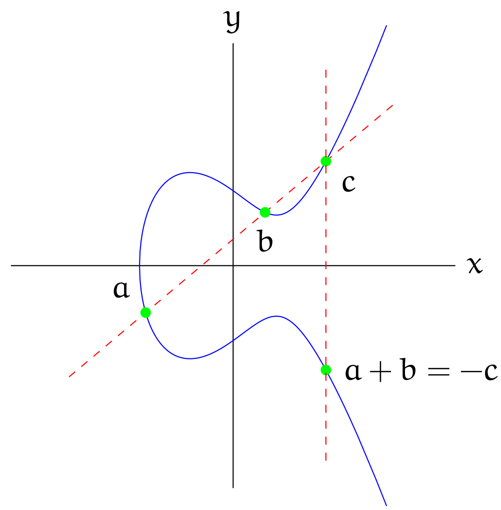

Figure 1.2: A "real" picture of the group law of an elliptic curve in action. Here, we have taken the identity element to be the point at infinity. Note that the identity element is itself a flex.

Before we move on to providing an overview of the rest of this thesis, we include one last corollary that will come handy in subsequent sections.

Corollary 17. *For each* $d \geq 4$*, the space of all curves of degree* d *that have a hyperflex forms a hypersurface in* $\mathbb{P}_k^{N(d)}$.

Proof. Recall notation from the proof of Proposition 14. The locus of curves in $\mathbb{P}_k^{N(d)}$ that have hyperflexes is the image X' of X under the projection map $X \to \mathbb{P}_k^{N(d)}$. The space $X'' \subset \mathbb{P}_k^{N(d)}$ of curves containing lines has too small dimension to be of relevance: note that the preimage of this space under the projection map $X \to \mathbb{P}_k^{N(d)}$ has dimension $N(d-1)+1$, which is necessarily less than $N(d)-1$. Hence, for the purpose of computing $\dim X'$, we can restrict our consideration to curves not containing lines. By Proposition 12, the fiber in X above a point in X' corresponding to a curve that does not contain a line is finite and hence has dimension 0. So, the preimage of $X' \setminus X''$ under the projection map $X \to \mathbb{P}_k^{N(d)}$ has dimension $0 + \dim X' = \dim X'$, but this dimension must be equal to $\dim X = N(d) - 1$. Thus, $\dim X' = N(d) - 1$, as desired. □

1.3 Overview and Main Results

We now present an overview of this thesis. In order to tackle Motivating Question 11, it will be necessary for us to acquire a basic familiarity with a number of powerful tools, namely the sheaves of principal parts, Chow rings, and Chern classes. The purpose of § 2 is to introduce and develop these tools, thereby providing an answer to Motivating Question 10.

With the necessary background material under our belts, we apply it to studying flexes and hyperflexes of plane curves in § 3. We start by using the constructions of the previous chapter (specifically, the Chern classes of the sheaves of principal parts) to rework the problem of counting flexes on plane curves in a more streamlined and motivated fashion. We then turn our attention to studying hyperflexes on plane curves. The astute reader would observe that this seems a bit hopeless: indeed, Corollary 15 states that a general plane curve of a specified degree does not have any hyperflexes, so what good would it be to attempt counting them? However, Corollary 17 tells us that the incidence correspondence parameterizing curves along with their hyperflex point-line pairs projects onto a hypersurface in the parameter space of curves. Therefore, although we do not expect *individual* curves to have hyperflexes, we do expect multi-parameter *families* of curves to have members with hyperflexes. This leads us to the following question.

Motivating Question 18. What can we say about inflection points in families of curves? What if the family contains a singular member?

The main results of this thesis (which are stated and proven in § 3) concern the last part of Motivating Question 18, about counting inflection points in families of curves acquiring singular members. What makes this question hard is that one cannot immediately apply the tools of § 2 to enumerating inflection points in such families; specifically, the sheaves of principal parts fail to be locally free at the singular points, so we cannot easily make sense of their Chern classes. To rectify this issue, we introduce a new system of locally free sheaves on one-parameter families of curves that serve to replace the sheaves of principal parts (see Theorem 68). The Chern classes of the new sheaves may be expressed as a main term coming from the inflection points plus error terms coming from the singular points; thus, to use these classes to answer enumerative questions about inflection points, we need to get a handle on the error terms.

Let $f \in k[[x, y]]$ be the analytic-local germ of an isolated plane curve singularity.

Our objective is to study the contribution of the singularity given by the vanishing of f in the analysis of inflectionary behavior in a family of smooth curves specializing to a curve with the prescribed singularity. Specifically, we seek to compute the error term $\mathrm{AD}^m(f)$ that arises when using the Chern classes of our replacement sheaves to study m^{th}-order inflection points in a family of curves. Taking $f = xy$, which corresponds to a **nodal singularity**, we explicitly compute

$$\mathrm{AD}^m(xy) = \binom{m+1}{4},$$

thereby recovering, by means of a direct and elementary argument, an enumerative formula due to Z. Ran (see Theorem 78). Then, we deduce as a corollary that

$$\mathrm{AD}^m(f) \geq \mu_f \cdot \binom{m+1}{4}$$

for an arbitrary singularity germ f; here, μ_f is the **Milnor number** of f and measures "how nodal" the singularity given by the vanishing of f is (see Corollary 90). The numerical function $\mathrm{AD}^m(f) - \mu_f \cdot \binom{m+1}{4}$ is an invariant of the isomorphism class of the singularity, and it measures the multiplicity with which the singularity counts as an m^{th}-order inflection point.

The subject of the fourth and final chapter of the thesis, § 4, is best described by the following question.

Motivating Question 19. Can we extend the notion of inflection point to make sense for more than just plane curves?

It turns out that the answer to Motivating Question 19 is yes; indeed, see Definition 98 for a more general notion of inflection point. Furthermore, the tools developed in § 2 along with the results of § 3 can be used to answer Motivating Question 11 in the context of this more general notion of inflection point. We illustrate how all of this works through a number of important examples and applications. Most notably, we apply the results of § 2 and § 3 to study **Weierstrass points** (a particular kind of inflection point) in families of curves acquiring singular members, and we calculate the divisors of weight-1 and weight-2 Weierstrass points of arbitrary order in the moduli space of curves (see Theorems 112 and and 113).

1.4 Setting up the Notation

In this brief section, we declare the primary notational conventions that we shall (unless otherwise specified) adhere to throughout this thesis.

- k denotes an algebraically closed field of characteristic 0.
- By "curve," we mean a 1-dimensional k-variety, and by "plane curve" we mean a 1-dimensional k-subvariety of the projective plane $\mathbb{P}^2_k$.
- All schemes are k-schemes, all varieties are k-varieties, and all points are k-valued.
- Script letters, like $\mathcal{L}$, $\mathcal{E}$, and $\mathcal{P}$, are used to denote sheaves; we reserve $\mathcal{L}$ for line bundles and $\mathcal{E}$ for arbitrary vector bundles. Non-script letters are typically used to denote modules.
- Let X be a scheme, let $p \in X$ be a point, let $V \hookrightarrow X$ be a closed subscheme, and let $U \subset X$ be an open subscheme. We denote by $\mathcal{I}_V$ the ideal sheaf of V (in particular, $\mathcal{I}_p$ is the ideal sheaf of p). For a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, we write:
  - $\Gamma(\mathcal{F})$ for the space of global sections of $\mathcal{F}$, viewed either as a module over global sections of $\mathcal{O}_X$ or as a k-vector space according to the context.
  - $\Gamma(U; \mathcal{F})$ for the space of sections of $\mathcal{F}$ defined on U.
  - $\mathcal{F}_p$ for the stalk of $\mathcal{F}$ at p.
  - $\mathcal{F}|_p$ for the fiber of $\mathcal{F}$ at p.
  - $\mathcal{F}|_U$ for the restriction of $\mathcal{F}$ to U.
  - $\mathcal{F}^\vee$ for the dual and $\mathcal{F}^{\vee\vee}$ for the double-dual of $\mathcal{F}$.
- Given a scheme X equipped with a map $\phi \colon X \to \mathbb{P}^r_k$ for some r, we denote by $\mathcal{O}_X(d)$ the pulled-back sheaf $\phi^*\mathcal{O}_{\mathbb{P}^r_k}(d)$.
- δ_{ij} denotes the Kronecker delta function, which is equal to 1 when $i = j$ and 0 when $i \neq j$.

Other notation used in the rest of this thesis will be defined as needed along the way.

Chapter 2

Three Fundamental Tools

> "No, no. These concepts were not dreamed up. They were natural and real."
>
> Shiing-Shen Chern, 1911–2004

In § 1.2.1, we discussed inflection points on plane curves from the perspective of intersection multiplicity, defining an m^{th}-order flex of a curve to be a point at which the tangent line has intersection multiplicity at least m with the curve. Subsequently, in § 1.2.2, we managed to find an explicit way of characterizing the flexes of a plane curve, but our strategy was rather *ad hoc* — indeed, it is not obvious how one can modify or generalize the argument involving the Hessian in Proposition 9 for the purpose of addressing more general problems like Motivating Questions 18 and 19. The primary objective of this chapter is to introduce and develop three exceedingly useful tools that we can use to study inflection points in a vastly more general and systematic fashion, thereby providing an answer to Motivating Question 10. Specifically, we shall discuss the following three tools:

(a) **Sheaves of principal parts**. In § 2.1, we demonstrate that inflection points of curves can be described as points at which sections of certain vector bundles, known as the sheaves of principal parts, become linearly dependent.

(b) **Chow rings**. In § 2.2, we introduce the Chow ring of a scheme X as a means of parameterizing closed subschemes of X in a way that conveniently captures relevant enumerative information about such subschemes.

(c) **Chern classes**. The loci on which sections of the sheaves of principal parts become linearly dependent are represented in the Chow ring by Chern classes; § 2.3 is devoted to defining Chern classes and proving properties of Chern classes that make them easy to compute.

So as not to lose sight of our ultimate motivation — namely, answering enumerative questions about inflection points — while detailing the above constructions, we shall frequently refer back to the case of studying flexes on plane curves.

Remark 20. Although we shall concentrate on applying the tools introduced in this chapter to solve enumerative questions about inflection points, these tools can be applied to solve a diverse collection of problems in enumerative geometry; see [EH16, Chapters 5–11] for a thorough discussion of these problems. Upon skimming the present chapter to become acquainted with the notation, the seasoned reader may wish to proceed directly to § 3, where we begin applying the tools developed in this chapter.

2.1 Sheaves of Principal Parts

The basic philosophy underlying our new tactic for studying inflection points is to reduce the difficult geometric problem of studying curves that have unusually large intersection multiplicity at a point with a line into a comparatively easier linear-algebraic problem. The standard way of accomplishing this reduction is to recast the original problem into a statement about vector bundles. As we shall soon see, the vector bundles that naturally arise in the context of studying inflection points of curves are known as the sheaves of principal parts. The purpose of the present section is to define these sheaves and prove some useful properties about them; familiarity with the workings of the sheaves of principal parts is crucial to understanding the main applications discussed in § 3 and § 4.

2.1.1 The Definition

Before stating the definition of the sheaves of principal parts, we demonstrate how they arise in the context of studying flexes on plane curves. For such a curve C, we are interested in describing its locus of flexes — the points at which the tangent line meets the curve with intersection multiplicity at least 3. In § 1.2.2, we used the Hessian to help us locate the flex *points*, but what if we instead tried to locate the flex *lines* (i.e., the tangent lines at the flex points)? Working from the perspective of the lines rather than the points has the happy advantage that we can rephrase our problem in linear-algebraic terms.

As it happens, the lines in the plane are precisely the "vanishing loci" of global sections of the line bundle $\mathcal{O}_{\mathbb{P}^2_k}(1)$. The following definition tells us what we mean by "vanishing locus" in this context.

Definition 21. Let X be a scheme, let $\mathcal{E}$ be a vector bundle on X, and let $\sigma \in \Gamma(\mathcal{E})$ be a global section. The **vanishing locus of** σ is a closed subscheme of X denoted by $V(\sigma)$ and defined as follows. If $U \subset X$ is an affine open subscheme on which the restriction of $\mathcal{E}$ is trivial, choose a basis $(e_1, \dots, e_{\operatorname{rk}\mathcal{E}})$ of $\Gamma(U; \mathcal{E})$ over $\Gamma(U; \mathcal{O}_X)$, and express the restriction of σ to U as $\sum_{i=1}^{\operatorname{rk}\mathcal{E}} a_i \cdot e_i$, where $a_i \in \Gamma(U; \mathcal{O}_X)$ for each i. Then $U \cap V(\sigma)$ is defined to be the closed subscheme of U cut out by the ideal $(a_1, \dots, a_{\operatorname{rk}\mathcal{E}}) \subset \Gamma(U; \mathcal{O}_X)$.

It is not too hard to show that $V(\sigma)$ is actually well-defined, in the sense that the affine-local descriptions provided by Definition 21 have the following two properties:

(a) They do not depend on the particular choices of local trivializations of $\mathcal{E}$; and

(b) They glue together in the appropriate fashion to form a closed subscheme of X.

Note that set-theoretically speaking, $V(\sigma)$ is the locus of points $p \in X$ at which the residue of σ is 0 (i.e., the image of σ in the fiber $\mathcal{E}|_p$ is 0).

With these formalities out of the way, let us see how Definition 21 applies in the case where $X = \mathbb{P}^2_k$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2_k}(1)$. Taking homogeneous coordinates $[X_1 : X_2 : X_3]$ on $\mathbb{P}^2_k$, recall that the space $\Gamma(\mathcal{O}_{\mathbb{P}^2_k}(1))$ may be identified with the space of homogeneous linear forms in the variables X_1, X_2, X_3. Let $U = \operatorname{Spec} k[x_1, x_2] \subset \mathbb{P}^2_k$ be the standard affine patch where $X_3 \neq 0$. Given a global section $\sigma = a \cdot X_1 + b \cdot X_2 + c \cdot X_3 \in \Gamma(\mathcal{O}_{\mathbb{P}^2_k}(1))$ where $a, b, c \in k$, the restriction map

$$\Gamma(\mathcal{O}_{\mathbb{P}^2_k}(1)) \to \Gamma(U; \mathcal{O}_{\mathbb{P}^2_k}(1)) = k[x_1, x_2]$$

sends σ to $a \cdot x_1 + b \cdot x_2 + c$. By Definition 21, it follows that

$$U \cap V(\sigma) = \operatorname{Spec} k[x_1, x_2]/(ax_1 + bx_2 + c),$$

so the vanishing locus of σ is a line on U. By computing the restriction of $V(\sigma)$ to the other two standard affine patches of $\mathbb{P}^2_k$, one readily verifies that $V(\sigma)$ is none other than the line with homogeneous equation $a \cdot X_1 + b \cdot X_2 + c \cdot X_3 = 0$.

The key takeaway from the lengthy digression above is that if we want to get at the flex lines of a curve, it may be productive to consider the k-vector space $\Gamma(\mathcal{O}_{\mathbb{P}^2_k}(1))$, which gives us a nice linear-algebraic parametrization of the lines in $\mathbb{P}^2_k$. Fixing a point $p \in C$ and taking any basis $(\sigma_1, \sigma_2, \sigma_3)$ of $\Gamma(\mathcal{O}_{\mathbb{P}^2_k}(1))$ (noting of course that the space of homogeneous linear forms in three variables is 3-dimensional), the condition that some line in the plane is a flex line for C at p is equivalent to the condition that there exists some scalars $a_1, a_2, a_3 \in k$ such that

$$\mathrm{mult}_p(C, V(\sigma)) \geq 3,$$

where $\sigma = a_1 \cdot \sigma_1 + a_2 \cdot \sigma_2 + a_3 \cdot \sigma_3$.

In the proof of Lemma 4, we showed that if L is a line, then $\mathrm{mult}_p(C, L)$ is equal to the "order of vanishing" of L along C at p, in the following (admittedly imprecise) sense: if we take the equation of L and plug it into the equation of C, the resulting polynomial vanishes to order precisely $\mathrm{mult}_p(C, L)$ at p. Thus, taking $\sigma \in \Gamma(\mathcal{O}_{\mathbb{P}^2_k}(1))$, we have that $V(\sigma)$ is a flex line for C at p if and only if $V(\sigma)$ has "order of vanishing" at least 3 along C at p. How can we make precise sense of this notion of "order of vanishing"? We can do it by means of the following three steps:

(a) Let C be embedded in $\mathbb{P}^2_k$ via a map $\iota \colon C \hookrightarrow \mathbb{P}^2_k$. To determine the behavior of the section σ along the curve C, we can pull σ back along ι, obtaining a global section $\iota^*\sigma$ of the pulled-back line bundle $\mathcal{O}_C(1)$.

(b) Next, to determine the behavior of $\iota^*\sigma$ at the point p up to third-order, we can consider the image $\tau_{\iota^*\sigma}$ of $\iota^*\sigma$ under the map of global sections

$$\Gamma(\mathcal{O}_C(1)) \to \Gamma(\mathcal{O}_C(1) \otimes \mathcal{O}_C/\mathcal{I}_p^3)$$

induced by the natural map of sheaves

$$\mathcal{O}_C(1) \to \mathcal{O}_C(1) \otimes \mathcal{O}_C/\mathcal{I}_p^3.$$

(c) It should then be the case that σ has order of vanishing at least 3 along C at p if and only if $\tau_{\iota^*\sigma}$ is the zero-section of the sheaf $\mathcal{O}_C(1) \otimes \mathcal{O}_C/\mathcal{I}_p^3$.

Based on the outline above, we state the following definition, which makes our intuitive notion of order of vanishing precise.

Definition 22. Let C be a curve, let Y be a scheme, and let $\iota\colon C \to Y$ be a morphism. Let $p \in C$ be a smooth point, let $\mathcal{E}$ be a vector bundle on Y, and let $\sigma \in \Gamma(\mathcal{E})$ be a global section. Then we say that σ **vanishes to order at least** m **along** C **at** p if the corresponding section $\tau_{\iota^*\sigma}$ of $\iota^*\mathcal{E} \otimes \mathcal{O}_C/\mathcal{I}_p^m$ is equal to the zero-section.

The next lemma demonstrates that the multiplicity of the vanishing locus of a section at a point is the same as its order of vanishing at the point. In terms of studying flexes on plane curves, this means that

$$\operatorname{mult}_p(C, V(\sigma)) = \operatorname{mult}_p(V(\iota^*\sigma)) \geq 3 \tag{2.1}$$

if and only if σ vanishes to order at least 3 along C at p.[1]

Lemma 23. *With notation as in Definition 22, we have* $\operatorname{mult}_p(V(\iota^*\sigma)) \geq m$ *if and only if* σ *vanishes to order at least* m *along* C *at* p.

Proof. Notice that the order of vanishing of σ along C at p is the same as the order of vanishing of $\iota^*\sigma$ along C at p. It therefore suffices to prove that if $\mathcal{E}$ is a vector bundle on C and $\sigma \in \Gamma(\mathcal{E})$ is a global section, then $\operatorname{mult}_p(V(\sigma)) \geq m$ if and only if the corresponding section τ_σ of $\mathcal{E} \otimes \mathcal{O}_C/\mathcal{I}_p^m$ is equal to the zero-section. Since $\mathcal{E} \otimes \mathcal{O}_C/\mathcal{I}_p^m$ is supported at p, we have that τ_σ is equal to the zero-section if and only if its germ $(\tau_\sigma)_p \in (\mathcal{E} \otimes \mathcal{O}_C/\mathcal{I}_p^m)_p$ is equal to 0. We can compute the stalk $(\mathcal{E} \otimes \mathcal{O}_C/\mathcal{I}_p^m)_p$ after passing to an open neighborhood of p on which $\mathcal{E}$ is trivial; doing so, we find that

$$(\mathcal{E} \otimes \mathcal{O}_C/\mathcal{I}_p^m)_p \simeq (\mathcal{O}_{C,p}/I_p^m)^{\oplus \operatorname{rk}\mathcal{E}}, \tag{2.2}$$

where $I_p \subset \mathcal{O}_{C,p}$ is the maximal ideal corresponding to the point p. Suppose that $(\tau_\sigma)_p \mapsto (a_1, \dots, a_{\operatorname{rk}\mathcal{E}})$ under the identification given by (2.2). Then the condition that $(\tau_\sigma)_p = 0$ is equivalent to the condition that $a_i \in I_p^m$ for each $i \in \{1, \dots, \operatorname{rk}\mathcal{E}\}$. Now, it follows from Definition 21 that

$$\operatorname{mult}_p V(\sigma) = \dim_k \mathcal{O}_{C,p}/(a_1, \dots, a_{\operatorname{rk}\mathcal{E}}),$$

[1] The equality of multiplicities in (2.1) follows from the definition of scheme-theoretic intersection.

so the condition that $a_i \in I_p^m$ for every i is equivalent to the condition that

$$\operatorname{mult}_p V(\sigma) \geq \dim_k \mathcal{O}_{C,p}/I_p^m = m,$$

because $\mathcal{O}_{C,p}$ is a discrete valuation ring (since p is a smooth point of C). Thus, we have the lemma. □

Armed with a rigorous notion of order of vanishing, let us return to the problem of studying flexes on the plane curve C. It follows from Lemma 23 that all we need to do to find its flex lines is to look at each smooth point $p \in C$ and check whether some linear combination of the basis sections $\sigma_1, \sigma_2, \sigma_3$ vanishes to order at least 3 at p. To do this, it would be convenient if we could somehow stitch the k-vector spaces $\Gamma(\mathcal{O}_C(1) \otimes \mathcal{O}_C/\mathcal{I}_p^3)$ together, so that we are not merely looking at one point at a time. This is exactly what the sheaves of principal parts are intended to accomplish.

As a last remark before we define the sheaves of principal parts, recall from Motivating Question 18 that we are not only interested in studying inflection points on individual curves, but also in "families" of curves. It will therefore be useful for us to introduce the sheaves of principal parts in the context of "families." The next definition specifies what we mean by the term "family."

Definition 24. Let X, B be smooth, irreducible varieties, and let $\pi\colon X \to B$ be a proper morphism with irreducible geometric fibers. We say that X/B is a **family** with **total space** X and **base** B.

At last, we are ready to define our first main tool, the sheaves of principal parts.[2]

Definition 25. Let X/B be a family, let $\Delta \hookrightarrow X \times_B X$ be the diagonal, which is a closed subscheme with ideal sheaf $\mathcal{I}_\Delta$, and let $\pi_1, \pi_2\colon X \times_B X \to X$ be the projection maps onto the left and right factors, respectively. The m^{th}**-order sheaf of (relative) principal parts** associated to a vector bundle $\mathcal{E}$ on X is a sheaf of $\mathcal{O}_X$-modules that is denoted by $\mathcal{P}^m_{X/B}(\mathcal{E})$ and defined by

$$\mathcal{P}^m_{X/B}(\mathcal{E}) = \pi_{1*}(\pi_2{}^*\mathcal{E} \otimes \mathcal{O}_{X \times_B X}/\mathcal{I}_\Delta^m).$$

[2]The sheaves of principal parts were first defined by A. Grothendieck in his seminal text [Gro67, IV, § 16]. Note that the phrase "principal parts," as used here, should not be confused with the phrase "principal parts" as used in complex analysis!

Remark 26. When the base space B is equal to $\operatorname{Spec} k$, so that the family X/B consists of a single fiber (as in the case of studying flexes on an individual plane curve), the sheaves of principal parts are simply denoted $\mathcal{P}^m_X(\mathcal{E})$.

The next proposition verifies that the sheaves of relative principal parts do their bidding, in the sense that they tell us when sections of a vector bundle vanish to a specified order at a point.

Proposition 27. *With notation as in Definition 25, for each point $p \in X$ we have the isomorphism of k-vector spaces*

$$\mathcal{P}^m_{X/B}(\mathcal{E})|_p \simeq \Gamma((\mathcal{E} \otimes \mathcal{O}_X/\mathcal{I}^m_p)|_{X_{\pi(p)}}),$$

where $X_{\pi(p)} = \pi(p) \times_B X$ denotes the fiber of the map $\pi \colon X \to B$ lying over the point $\pi(p) \in B$.

Proof. Let $p \in X$. We shall simplify matters by working not on all of X but on an affine open neighborhood of p; to justify this simplification, we must show that the relevant constructions commute with restricting to such a neighborhood.

On the one hand, notice that $(\mathcal{E} \otimes \mathcal{O}_X/\mathcal{I}^m_p)|_{X_{\pi(p)}}$ is a skyscraper sheaf supported at p, so it can certainly be computed affine-locally. It is a more intricate matter to show that the construction of the fiber

$$\mathcal{P}^m_{X/B}(\mathcal{E})|_p = \mathcal{P}^m_{X/B}(\mathcal{E})_p \otimes_{\mathcal{O}_{X,p}} \kappa(p)$$

commutes with restriction to an affine open neighborhood of p. To see this, take affine open subschemes $U' \subset B$ containing $\pi(p)$ and $U \subset \pi^{-1}(U') \subset X$ containing p; we must show that

$$\mathcal{P}^m_{X/B}(\mathcal{E})|_U \simeq \mathcal{P}^m_{U/U'}(\mathcal{E}|_U) \tag{2.3}$$

as sheaves on U. Since pushforwards commute with pullbacks along open embeddings, we have that

$$\mathcal{P}^m_{X/B}(\mathcal{E})|_U \simeq \pi_{1*}((\pi_2{}^*\mathcal{E} \otimes \mathcal{O}_{X \times_B X}/\mathcal{I}^m_\Delta)|_{U \times_B X}), \tag{2.4}$$

Moreover, because $\mathcal{O}_{X \times_B X}/\mathcal{I}^m_\Delta$ is supported only on points of Δ, we have

$$\pi_{1*}((\pi_2{}^*\mathcal{E} \otimes \mathcal{O}_{X \times_B X}/\mathcal{I}^m_\Delta)|_{U \times_B X}) = \pi_{1*}((\pi_2{}^*\mathcal{E} \otimes \mathcal{O}_{X \times_B X}/\mathcal{I}^m_\Delta)|_{U \times_{U'} U}). \tag{2.5}$$

Since the sheafification functors for tensor product and pullback both commute with restriction to an open subscheme, we have

$$(\pi_2{}^*\mathcal{E} \otimes \mathcal{O}_{X\times_B X}/\mathcal{J}_\Delta^m)|_{U\times_{U'}U} = \pi_2{}^*(\mathcal{E}|_U) \otimes \mathcal{O}_{U\times_{U'}U}/\mathcal{J}_{\Delta'}^m, \tag{2.6}$$

where Δ' denotes the pullback of Δ to the open subscheme $U \times_{U'} U \subset X \times_B X$. Then (2.3) follows upon combining the results of (2.4), (2.5), and (2.6).

Now, write $U = \operatorname{Spec} R$ and $U' = \operatorname{Spec} S$. Then $U \times_{U'} U = \operatorname{Spec}(R \otimes_S R)$, and the ideal I cutting out the diagonal Δ' in $U \times_{U'} U$ is the kernel of the obvious multiplication map $R \otimes_S R \to R$. In particular, choosing a collection $\{x_i\} \subset R$ of elements that generate R as an S-algebra, we have that $I = (\{x_i \otimes 1 - 1 \otimes x_i\})$.[3] Writing $E = \Gamma(U; \mathcal{E})$, we have

$$\begin{aligned}\Gamma\big(U \times_{U'} U; \pi_2{}^*(\mathcal{E}|_U) \otimes \mathcal{O}_{U\times_{U'}U}/\mathcal{J}_{\Delta'}^m\big) &= (R \otimes_S E) \otimes_{(R\otimes_S R)} (R \otimes_S R)/I^m \\ &= (R \otimes_S E)/I^m(R \otimes_S E),\end{aligned}$$

where we are viewing the last expression above as a module over $R \otimes_S R$ with the obvious scalar multiplication. Pushing forward along π_1, we find

$$\Gamma\big(U; \pi_{1*}(\pi_2{}^*(\mathcal{E}|_U) \otimes \mathcal{O}_{U\times_{U'}U}/\mathcal{J}_{\Delta'}^m)\big) = (R \otimes_S E)/I^m(R \otimes_S E), \tag{2.7}$$

where we are now viewing the right-hand side above as an R-module, with scalar multiplication given by $a \cdot (r \otimes e) = (a \otimes 1) \cdot (r \otimes e) = (ar) \otimes e$. Then, localizing at p and tensoring with $\kappa(p)$ yields that the fiber of $\mathcal{P}_{X/B}^m(\mathcal{E})$ at p is given by

$$\begin{aligned}\mathcal{P}_{X/B}^m(\mathcal{E})|_p &= \mathcal{P}_{X/B}^m(\mathcal{E})_p \otimes_{\mathcal{O}_{X,p}} \kappa(p) \\ &= \big([(R \otimes_S E)/I^m(R \otimes_S E)] \otimes_R R_p\big) \otimes_{R_p} \kappa(p) \\ &= (\kappa(p) \otimes_S E)/I^m(\kappa(p) \otimes_S E).\end{aligned}$$

Now let $a_i \in k$ be the "coordinates" of p (by this, we mean that the maximal ideal of R corresponding to p can be expressed as $(\{a_i - x_i\})$). Then I, viewed as an ideal of $\kappa(p) \otimes_S R$, can be expressed as $I = (\{a_i \otimes 1 - 1 \otimes x_i\})$. But since $a_i \in k$ and S is a k-algebra, we have that $a_i \otimes 1 - 1 \otimes x_i = 1 \otimes (a_i - x_i)$, so we deduce that

$$\mathcal{P}_{X/B}^m(\mathcal{E})|_p = (\kappa(p) \otimes_S E)/(\{1 \otimes (a_i - x_i)\})^m(\kappa(p) \otimes_S E).$$

[3]Note that the collection of generators $\{x_i\}$ may be taken to be finite, because the morphism $X \to B$ was stipulated to be proper, and hence of finite type.

On the other hand, we have that $U_{\pi(p)} = \pi(p) \times_{U'} U = \operatorname{Spec} \kappa(p) \otimes_S R$. It follows that

$$\begin{aligned}
\Gamma((\mathcal{E}|_U \otimes \mathcal{O}_U/\mathcal{I}_p^m)|_{U_{\pi(p)}}) &= \kappa(p) \otimes_S \Gamma(\mathcal{E}|_U \otimes \mathcal{O}_U/\mathcal{I}_p^m) \\
&= \kappa(p) \otimes_S [E/I_p^m E] \\
&= (\kappa(p) \otimes_S E)/(\{1 \otimes (a_i - x_i)\})^m(\kappa(p) \otimes_S E),
\end{aligned}$$

which is precisely the expression that we obtained for the fiber. □

Recall that in the case of studying flexes on a plane curve C, we constructed a map that assigned to a section $\sigma \in \Gamma(\mathcal{O}_{\mathbb{P}_k^2}(1))$ a corresponding section $\tau_{\iota^*\sigma} \in \Gamma(\mathcal{O}_C(1) \otimes \mathcal{O}_C/\mathcal{I}_p^m)$. In Proposition 27, we demonstrated that the k-vector spaces $\Gamma(\mathcal{O}_C(1) \otimes \mathcal{O}_C/\mathcal{I}_p^m)$ fit together to form a sheaf over C, namely the sheaf of principal parts $\mathcal{P}_C^m(\mathcal{O}_{\mathbb{P}_k^2}(1))$, so it is natural to wonder whether the sections $\tau_{\iota^*\sigma}$ fit together to form a section of this sheaf. The following lemma answers this question in the affirmative.

Lemma 28. *With notation as in Definition 25, there is a natural assignment*

$$\sigma \in \Gamma(\mathcal{E}) \mapsto \tau_\sigma \in \Gamma(\mathcal{P}_{X/B}^m(\mathcal{E}))$$

with the property that the vanishing locus of τ_σ contains the point $p \in X$ precisely when σ vanishes to order m along the fiber $X_{\pi(p)}$.[4]

Proof. Let $p \in X$, and let $\sigma \in \Gamma(\mathcal{E})$. We shall define τ_σ affine-locally; it is not too hard to check that the resulting local sections glue together in the appropriate way. Consider the affine open set U defined in the proof of Proposition 27. We define τ_σ on U to be given by the tensor

$$1 \otimes e \in (R \otimes_S E)/I^m(R \otimes_S E) = \Gamma(U; \mathcal{P}_{X/B}^m(\mathcal{E})).$$

Now that we have defined τ_σ, the rest of the lemma is tautological. Indeed, observe that the residue (i.e., value) of τ_σ at p is equal, by Proposition 27, to what we previously called τ_σ, namely the image of σ in $\Gamma((\mathcal{E} \otimes \mathcal{O}_X/\mathcal{I}_p^m)|_{X_{\pi(p)}})$, and by Definition 22, this image is 0 if and only if σ vanishes to order m along $X_{\pi(p)}$ at p. □

[4]Note here that we are abusing notation by using the symbol τ_σ to denote the section formed by stitching together the objects that we used to call τ_σ in the proof of Lemma 23. This choice of notation is less cumbersome, if not less confusing!

2.1.2 When are Principal Parts Sheaves Locally Free?

Our aim was to reduce the challenging algebro-geometric problem of counting inflection points into an easier problem about vector bundles, but all we have done so far is construct a class of sheaves. In this regard, it is natural to wonder what conditions we can put on a family X/B so that the sheaves of principal parts associated to X/B end up being locally free.

The following highly useful proposition tells us that the sheaves of principal parts fit into nice exact sequences that allow us to prove properties about them by inducting on the order m. The subsequent corollary uses these exact sequences to show that the sheaves of principal parts are locally free when the family X/B is smooth.

Proposition 29. *Retain the setting of Definition 25. For each integer $m \geq 2$, we have the following right-exact sequence:*

$$\mathcal{E} \otimes (\mathrm{Sym}^{m-1} \Omega^1_{X/B}) \longrightarrow \mathcal{P}^m_{X/B}(\mathcal{E}) \longrightarrow \mathcal{P}^{m-1}_{X/B}(\mathcal{E}) \longrightarrow 0$$

*where $\Omega^1_{X/B} = \pi_{1*}(\mathcal{I}_\Delta/\mathcal{I}^2_\Delta)$ denotes the **sheaf of relative differentials** associated to the family X/B. Moreover, the above sequence is exact when X/B is smooth.*

Proof. We start with the short exact sequence of $\mathcal{O}_{X\times_B X}$-modules

$$0 \longrightarrow \mathcal{I}^{m-1}_\Delta/\mathcal{I}^m_\Delta \longrightarrow \mathcal{O}_{X\times_B X}/\mathcal{I}^m_\Delta \longrightarrow \mathcal{O}_{X\times_B X}/\mathcal{I}^{m-1}_\Delta \longrightarrow 0$$

Since tensoring with a vector bundle is flat, we deduce that

$$0 \longrightarrow \pi_2^*\mathcal{E} \otimes \mathcal{I}^{m-1}_\Delta/\mathcal{I}^m_\Delta \longrightarrow \pi_2^*\mathcal{E} \otimes \mathcal{O}_{X\times_B X}/\mathcal{I}^m_\Delta \longrightarrow \pi_2^*\mathcal{E} \otimes \mathcal{O}_{X\times_B X}/\mathcal{I}^{m-1}_\Delta \longrightarrow 0$$

Moreover, because the sheaves in the above sequence are supported on Δ, we can swap $\pi_2^*\mathcal{E}$ with $\pi_1^*\mathcal{E}$ in the above sequence, so we find that

$$0 \longrightarrow \pi_1^*\mathcal{E} \otimes \mathcal{I}^{m-1}_\Delta/\mathcal{I}^m_\Delta \longrightarrow \pi_2^*\mathcal{E} \otimes \mathcal{O}_{X\times_B X}/\mathcal{I}^m_\Delta \longrightarrow \pi_2^*\mathcal{E} \otimes \mathcal{O}_{X\times_B X}/\mathcal{I}^{m-1}_\Delta \longrightarrow 0$$

Pushing forward along π_1, we obtain the exact sequence

$$0 \longrightarrow \pi_{1*}(\pi_1^*\mathcal{E} \otimes \mathcal{I}^{m-1}_\Delta/\mathcal{I}^m_\Delta) \longrightarrow \mathcal{P}^m_{X/B}(\mathcal{E}) \longrightarrow \mathcal{P}^{m-1}_{X/B}(\mathcal{E}) \longrightarrow R^1\pi_{1*}(\mathcal{I}^{m-1}_\Delta/\mathcal{I}^m_\Delta)$$

where the last term above is the first-degree higher direct image of $\mathcal{I}_\Delta^{m-1}/\mathcal{I}_\Delta^m$. It turns out that $R^1\pi_{1*}(\mathcal{I}_\Delta^{m-1}/\mathcal{I}_\Delta^m) = 0$ because the sheaves in the above sequence are supported on Δ, and the map π_1 is an isomorphism from Δ. By the push-pull formula, we have

$$\pi_{1*}(\pi_1^*\mathcal{E} \otimes \mathcal{I}_\Delta^{m-1}/\mathcal{I}_\Delta^m) \simeq \mathcal{E} \otimes \pi_{1*}(\mathcal{I}_\Delta^{m-1}/\mathcal{I}_\Delta^m),$$

so to prove the proposition, it remains to show that

$$\operatorname{Sym}^{m-1}\Omega^1_{X/B} = \operatorname{Sym}^{m-1}(\pi_{1*}(\mathcal{I}_\Delta/\mathcal{I}_\Delta^2)) \simeq \pi_{1*}(\mathcal{I}_\Delta^{m-1}/\mathcal{I}_\Delta^m),$$

and this follows from the assumption that $\pi\colon X \to B$ is smooth; see [EH16, proof of Theorem 7.2 (d)] for one possible argument. □

Corollary 30. *Retain the setting of Definition 25. When the family* X/B *is smooth,* $\mathcal{P}^m_{X/B}(\mathcal{E})$ *is locally free of rank* m.

Proof. Since X/B is smooth, Proposition 29 tells us that we have the short exact sequences for each integer $m \geq 2$:

$$0 \longrightarrow \mathcal{E} \otimes (\operatorname{Sym}^{m-1}\Omega^1_{X/B}) \longrightarrow \mathcal{P}^m_{X/B}(\mathcal{E}) \longrightarrow \mathcal{P}^{m-1}_{X/B}(\mathcal{E}) \longrightarrow 0$$

Observe that the sheaf $\Omega^1_{X/B}$, and hence the sheaf $\mathcal{E} \otimes (\operatorname{Sym}^{m-1}\Omega^1_{X/B})$, is locally free because X/B is smooth. The proposition then follows immediately by inductively applying the following fact to the short exact sequences above: if $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ are sheaves of $\mathcal{O}_X$-modules that fit into a short exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

then $\mathcal{F}_2$ is locally free if $\mathcal{F}_1, \mathcal{F}_3$ are locally free. □

To see that the sheaves of principal parts fail to be locally free at singular points of the fibers of our family X/B, consider the following example, in which we compute the ranks of the sheaves of principal parts at the node of a nodal curve.

Example 31. Let $B = \operatorname{Spec} k$, let $X = C \subset \mathbb{P}^2_k$ be an irreducible plane curve with a nodal singularity at a point $p \in C$, and let $\mathcal{E} = \mathcal{L}$ be a line bundle. The rank of $\mathcal{P}^m_C(\mathcal{L})$ at p is given by the k-vector space dimension of the fiber $\mathcal{P}^m_C(\mathcal{L})|_p$; we worked out what the fiber is in Proposition 27, where we showed that

$$\mathcal{P}^m_C(\mathcal{L})|_p = \Gamma(\mathcal{L} \otimes \mathcal{O}_C/\mathcal{I}^m_p). \tag{2.8}$$

The dimensions of the spaces in (2.8) can be computed by passing to an open subscheme of C on which $\mathcal{L}$ is locally free, so we deduce that

$$\dim_k \mathcal{P}^m_C(\mathcal{L})|_p = \dim_k \mathcal{O}_{C,p}/I^m_p,$$

where $I_p \subset \mathcal{O}_{C,p}$ denotes the maximal ideal corresponding to p. This dimension is unchanged under taking the completion, so we have

$$\dim_k \mathcal{P}^m_C(\mathcal{L})|_p = \dim_k \widehat{\mathcal{O}}_{C,p}/I^m_p.$$

Now, by the definition of a nodal singularity, we have that $\widehat{\mathcal{O}}_{C,p} \simeq k[[x,y]]/(xy)$; under this identification, the ideal I_p corresponds to the ideal (x,y). It follows that

$$\dim_k \mathcal{P}^m_C(\mathcal{L})|_p = \dim_k k[[x,y]]/(xy,(x,y)^m) = 2m-1.$$

Thus, we conclude that the rank of $\mathcal{P}^m_C(\mathcal{L})$ at p is $2m-1$; for $m > 1$, this result is strictly greater than m, which is the rank of $\mathcal{P}^m_C(\mathcal{L})$ at a smooth point (see Corollary 30).

2.1.3 Where to Go Next?

Now that we have defined and proven the basic properties of the sheaves of principal parts, we need to figure out how to use them. Recall that in the example of flexes on a plane curve C, letting $(\sigma_1, \sigma_2, \sigma_3)$ be a basis of $\Gamma(\mathcal{O}_{\mathbb{P}^2_k}(1))$, we are interested in points $p \in C$ at which there exists a linear combination of the sections $\sigma_1, \sigma_2, \sigma_3$ vanishing to order at least 3 along C at p. We have shown that this condition is equivalent to the condition that there exists a linear combination of the sections $\tau_{\sigma_1}, \tau_{\sigma_2}, \tau_{\sigma_3} \in \Gamma(\mathcal{P}^3_C(\mathcal{O}_C(1)))$ vanishing at p. The set of such points p is by definition the vanishing locus of the global section $\sigma_1 \wedge \sigma_2 \wedge \sigma_3$ of the exterior power $\Lambda^3(\mathcal{P}^3_C(\mathcal{O}_C(1)))$. This vanishing locus has a special name.

Definition 32. Let $\mathcal{E}$ be a vector bundle on a scheme X, and let $\sigma_1, \dots, \sigma_n \in \Gamma(\mathcal{E})$ be global sections of $\mathcal{E}$. The **degeneracy locus** of $\sigma_1, \dots, \sigma_n$ is defined to be the vanishing locus of the global section $\sigma_1 \wedge \cdots \wedge \sigma_n \in \Gamma(\Lambda^n \mathcal{E})$.

With the view of addressing Motivating Question 11, we are not really interested in obtaining an explicit description of the degeneracy locus of the sections $\sigma_1, \sigma_2, \sigma_3$; rather, we want away to characterize the number of points at which this degeneracy locus is supported. In what follows, we introduce tools that allow us to provide useful enumerative characterizations of degeneracy loci of sections of vector bundles.

2.2 Chow Rings

The objective of this section is to introduce the theory of Chow rings, which provides us with a means of describing closed subschemes of a scheme in such a way that enough information is retained to be able to answer enumerative questions about those subschemes.[5] Although we will eventually restrict our attention to studying the closed subschemes that arise as degeneracy loci of sections of a vector bundle on a family of curves, it is useful and enlightening to have a taste of the general theory.

2.2.1 Classifying Closed Subschemes $\mathbb{P}^2_k$

We begin our discussion of Chow rings with an illustrative example: classifying the closed subschemes of $\mathbb{P}^2_k$. The goal is to obtain a classification that not only keeps track of the multiplicities of closed subschemes, but also serves as a computationally useful tool for studying intersections thereof. In keeping with the general mantra[6] that "unions are sums" and "intersections are products," it is natural to hope that our classification can be equipped with additive and multiplicative structures (which is precisely what we need to construct a ring) corresponding to the operations of union and intersection.

For now, let us restrict our consideration to pure-dimensional closed subschemes $X \subset \mathbb{P}^2_k$. Any such X is either 0-, 1-, or 2-dimensional, so let us study each of these cases separately. The only 2-dimensional closed subscheme of $\mathbb{P}^2_k$ is $\mathbb{P}^2_k$ itself, and the intersection between $\mathbb{P}^2_k$ and any subscheme $X \subset \mathbb{P}^2_k$ is obviously just X. Thus, there is not much to learn from studying the 2-dimensional case, except for the following trivial but important observation: $\mathbb{P}^2_k$ serves as a sort of "identity element" for the operation

[5]Historical aside: The Chow ring is named in honor of intersection theorist W.-L. Chow who studied equivalence classes of closed subvarieties of a variety (see [Cho56]).

[6]Note that this concept is ubiquitous in mathematics, arising in fields like probability theory, category theory, and elsewhere.

of intersection on the collection of closed subschemes of $\mathbb{P}^2_k$. For labeling purposes, let C_2 denote the (singleton) class of 2-dimensional closed subschemes of $\mathbb{P}^2_k$.

Next, consider the case where X is 0-dimensional. Any such X is supported at finitely many points $p \in \mathbb{P}^2_k$, and the sum of the multiplicities $\text{mult}_p X$ of X at the points p in its support is a positive integer called the **length** of X. Note that the general 0-dimensional closed subscheme fails to meet any given closed subscheme of $\mathbb{P}^2_k$ other than $\mathbb{P}^2_k$ itself. Thus, if we take the empty set to serve as the zero element of our ring and if we denote by $C_0(j)$ the class of length-j subschemes, then the classes $C_0(j)$ act as annihilators, because their intersection/product with any other class is the class of the empty set.

It remains to treat the case where X is 1-dimensional. As far as intersections between closed subschemes $X, Y \subset \mathbb{P}^2_k$ are concerned, this is the only non-obvious case. Furthermore, if X and Y are chosen sufficiently generally, they do not share an irreducible component, implying that that their intersection is 0-dimensional.[7] Thus, we want the classes corresponding to X and Y to have the property that their product is the class associated to the length of their intersection. One of the most important properties of a 1-dimensional closed subscheme X of the plane is its **degree**, which is defined as the degree of the homogeneous polynomial whose vanishing locus is X. What does the degree have to do with intersections? For starters, observe that the degree has the following equivalent description: the degree of X is the length of the intersection between X and any line. This description is the easiest case of a far more profound result called Bézout's Theorem, which is stated as follows.[8]

Theorem 33 (Bézout's Theorem)**.** *Let $C, D \subset \mathbb{P}^2_k$ be plane curves of degrees c and d, respectively. Then $C \cap D$ is a 0-dimensional closed subscheme of $\mathbb{P}^2_k$ with length cd if and only if C and D do not share an irreducible component.*

Proof. We shall not provide a complete proof of Bézout's Theorem here, because it is lengthy and not immediately relevant to the progression of this thesis.

There are multiple different proofs of the theorem. The classical approach is reminiscent of the argument involving the Hessian that we used to tackle flexes on plane curves: an *ad hoc* construction called the **resultant scheme** is introduced that keeps

[7]This assertion is an easy corollary of the classical proof of Bézout's Theorem involving the resultant scheme.

[8]Historical aside: Bézout's Theorem was first properly stated by I. Newton in his *Principia Mathematica* (see [New72, Lemma 28]), and later by Étienne Bézout (see [B06]).

track of the points of intersection of two plane curves, along with the intersection multiplicities at each point. Geometrically speaking, the resultant scheme is the image in $\mathbb{P}^1_k$ of the scheme-theoretic intersection $C \cap D$ under the projection map away from a fixed point $p \in \mathbb{P}^2_k$ not lying on C or on D.

For a more modern approach, refer to [Har95, Exercise 13.17] or [Vak17, Exercise 18.6.K] for an argument involving Hilbert functions. □

Bézout's Theorem suggests that we ought to classify the 1-dimensional closed subschemes of X by degree, so let $C_1(j)$ denote the class corresponding to degree j. The classes we have constructed obey the following rules for multiplication: for any $0 \leq i \leq 1$ and positive integers j and j', we have that

(a) the 2-dimensional class C_2 is the multiplicative identity; i.e., $C_2 \cdot C_2 = C_2$ and $C_2 \cdot C_i(j) = C_i(j) \cdot C_2$;

(b) the 0-dimensional classes $C_0(j)$ are annihilators; i.e., $C_0(j) \cdot C_i(j') = 0$; and

(c) the 1-dimensional classes $C_1(j)$ multiply to give 0-dimensional classes in accordance with Bézout's Theorem; i.e., $C_1(j) \cdot C_1(j') = C_0(jj')$.

As for addition, if unions are supposed to be like sums, then a reducible closed subscheme should be expressible as the sum of its irreducible components. In particular, for any $0 \leq i \leq 1$ and positive integers j and j', we have that

$$C_i(j) + C_i(j') = C_i(j + j').$$

It is easy to see that these multiplication and addition rules satisfy the laws, like commutativity, associativity, and distributivity, that are necessary to define a ring; however, the classes defined above do not *a priori* form a ring. As we shall see in the next section, § 2.2.2, they are generators of what is called the Chow ring of $\mathbb{P}^2_k$.

2.2.2 Defining the Chow Ring

In this section, we discuss the steps required to define the Chow ring. The first step is to give precise meaning to the notion of "classes" of subschemes introduced in § 2.2.1. Roughly speaking, the following definition provides us with a structure on the closed subschemes of X that allows us to add them together with multiplicities.

Definition 34. Let X be a scheme, and let $Z(X)$ denote the free abelian group on the collection of integral closed subschemes of X. The elements of $Z(X)$ are called **cycles**.

We next define what it means for two cycles to belong to the same "class." The basic idea is that two cycles are in the same class if there is a way to get from one to the other via a sequence of families of cycles, each of which is parameterized by $\mathbb{P}^1_k$.

Definition 35. Let X be a scheme, and let $\Phi \subset X \times_k \mathbb{P}^1_k$ be an irreducible subvariety with the property that the restriction of the projection map $\pi_2 \colon X \times_k \mathbb{P}^1_k \to \mathbb{P}^1_k$ to Φ is dominant. For each point $t \in \mathbb{P}^1_k$, let $\Phi_t \subset X$ be the image of $\Phi \cap X \times \{t\}$ under the projection map $\pi_1 \colon X \times_k \mathbb{P}^1_k \to X$. Now, consider the subgroup $\mathrm{Rat}(X) \subset Z(X)$ generated by cycles of the form $\Phi_t - \Phi_{t'}$ for any Φ as above and points $t, t' \in \mathbb{P}^1_k$. We say that two cycles $A, B \in Z(X)$ are **rationally equivalent** if $A - B \in \mathrm{Rat}(X)$.

The notion of rational equivalence, as defined above, evidently gives an equivalence relation on the abelian group $Z(X)$, as the reflexivity, symmetry, and transitivity properties are all trivially satisfied.

Remark 36. One might protest that the transitivity property has in some sense been artificially engineered into Definition 35, by allowing the length of the sequence of interpolating cycles to be arbitrary. Indeed, it is natural to ask the following question: given two rationally equivalent cycles A and B on a scheme X, does there necessarily exist a cycle Φ on $X \times_k \mathbb{P}^1_k$ interpolating between A and B? As it happens, the answer is no; easy counterexamples abound in the case where X is not integral, but problems arise even in what might be considered the simplest non-planar case. Indeed, let $X = \mathbb{P}^3_k$, let A be the cycle corresponding to a twisted cubic curve in $\mathbb{P}^3_k$ (i.e., the image of the Veronese embedding of $\mathbb{P}^1_k$ in $\mathbb{P}^3_k$), and let B be a smooth plane cubic with an embedded point. Then it turns out that the cycles A, B are rationally equivalent, but there is no single family over $\mathbb{P}^1_k$ parameterizing between them. Indeed, by [PS85, Theorem], the Hilbert scheme parameterizing subschemes of $\mathbb{P}^3_k$ with Hilbert polynomial $3m + 1$ has two irreducible components, one of which contains twisted cubics in its interior and the other of which contains smooth plane cubics with an embedded point in its interior. It is therefore impossible for any single $\mathbb{P}^1_k$ to "link" A and B.

We can now make sense of the vague term "class" used above: two cycles A, B belong to the same "class" if and only if A, B are rationally equivalent. In other words, the "classes" are precisely the equivalence classes of $Z(X)$ under rational equivalence, which are the same as the elements of the quotient group $Z(X)/\operatorname{Rat}(X)$. This leads to the definition of the Chow group.

Definition 37. Let X be a scheme. The group $Z(X)/\operatorname{Rat}(X)$ of cycles modulo rational equivalence is called the **Chow group** of X and is denoted by $A(X)$.

Given a cycle $A \in Z(X)$, the image of A under the quotient map $Z(X) \to A(X)$ is denoted by $[A]$ and is called the **class** of A.

The next step on our journey to defining the Chow ring is to consider the dimensions of the closed subschemes of X. Recall from our analysis of closed subschemes of $\mathbb{P}^2_k$ in § 2.2.1 that we could categorize classes based on the dimensions of the subschemes that represent them. The next proposition makes this notion precise.

Proposition 38. *Let X be a scheme, and for each integer $k \in \{0, \dots, \dim X\}$, let $Z_k(X) \subset Z(X)$ be the subgroup of k-**cycles**, i.e., cycles all of whose terms have dimension k. The Chow group $A(X)$ is graded by dimension, in the sense that*

$$A(X) \simeq \bigoplus_{k=0}^{\dim X} A_k(X),$$

where $A_k(X)$ is the image of the subgroup $Z_k(X)$ under the quotient map $Z(X) \to A(X)$.

Proof. Observe that $Z(X)$ is evidently graded by dimension, in the sense that

$$Z(X) \simeq \bigoplus_{k=0}^{\dim X} Z_k(X).$$

To prove that $A(X)$ is likewise graded by dimension, it suffices to show that rational equivalence respects the grading of $Z(X)$; i.e., it suffices to show that if a cycle $A \in Z(X)$ is rationally equivalent to a k-cycle $B \in Z_k(X)$, then A is necessarily also a k-cycle. But since $\operatorname{Rat}(X)$ is generated by rational equivalences between honest subschemes (rather than cycles) of X, it further suffices to treat the case where $A, B \hookrightarrow X$ are closed subschemes that are "linked" by a single $\mathbb{P}^1_k$. Let $\Phi \subset X \times_k \mathbb{P}^1_k$ be the irreducible subvariety realizing the rational equivalence between A and B. Then the fiber above any point in $\mathbb{P}^1_k$ along the projection map $\pi_2 \colon X \times_k \mathbb{P}^1_k \to \mathbb{P}^1_k$ necessarily has codimension 1 in Φ. Since A and B are both fibers of this map, it follows that $\dim A = (\dim \Phi) - 1 = \dim B$, which is the desired result. □

Remark 39. Suppose X is a variety of pure dimension. When it is convenient, we write the graded components of the Chow ring in terms of their codimension rather than dimension; i.e., we write $A^k(X)$ for $A_{\dim X - k}(X)$.

The third and final step toward defining the Chow ring is making sense of how intersections can be turned into products. Our rudimentary analysis in § 2.2.1 suggests that we should be able to define a multiplicative structure on the Chow group $A(X)$ in such a way that the product of cycles corresponds to their intersection. Accomplishing this is far from easy: as stated in [EH16, § 1.2.3], "a great deal of the development of algebraic geometry over the past two-hundred years is bound up in the attempt to [discover] precise notions of the sense in which intersection of subvarieties resembles multiplication." In fact, it was not until three decades ago, when W. Fulton published his masterpiece [Ful98], that a completely rigorous and watertight notion of "intersection product" was discovered.

We shall omit the proof of the existence of the Chow ring because it would be a monumental task to reproduce (indeed, it is one of the primary objectives of Fulton's lengthy text [Ful98]). In order to state the theorem that guarantees the existence of the Chow ring, we require the following definition, which basically tells us when two subvarieties intersect "nicely".

Definition 40. Let X be a variety, and let $A, B \hookrightarrow X$ be subvarieties. For a point $p \in A \cap B$ at which all three of A, B, X are smooth, we say that A and B **intersect transversely** at p if the tangent spaces of A and B at p span the tangent space of X at p. We say that A and B **intersect generically transversely** if they intersect transversely at a general point of each irreducible component of their intersection $A \cap B$.

At long last, we are in position to introduce the Chow ring.

Theorem 41. *Let X be a smooth, irreducible, quasi-projective variety.*

(a) ("Moving Lemma") For any cycles $A, B \in Z(X)$, there exist cycles $A', B' \in Z(X)$ with $[A] = [A']$ and $[B] = [B']$ such that A' and B' meet generically transversely.

*(b) There is a unique way of endowing the Chow group $A(X)$ with a multiplication operation $\cdot\colon A^i(X) \times A^j(X) \to A^{i+j}(X)$, called the **intersection product**, that satisfies the following property: if $A, B \subset X$ are irreducible subvarieties that meet generically transversely, then $[A] \cdot [B] = [A \cap B]$. This multiplicative structure makes the graded group $A(X)$ into a commutative ring, called the **Chow ring** of X.*

2.2.3 Basic Properties and Examples

In this section, we discuss some basic properties of Chow rings, and we briefly describe the Chow ring of projective space. We shall not provide detailed proofs of results stated here, because they are very well-exposited and described in [EH16, § 1–2].

Functoriality: Pushforwards and Pullbacks

In subsequent sections, we shall require an understanding of how Chow groups "talk to each other." More precisely, given a morphism of schemes $X \to Y$, we need to figure out how classes in $A(X)$ push forward to classes in $A(Y)$ and how classes in $A(Y)$ pull back to classes in $A(X)$. In the following proposition, we define the pushforward of Chow groups.

Proposition 42. *Let $\phi\colon X \to Y$ be a proper morphism of schemes, and for a subvariety $X' \hookrightarrow X$, consider the assignment $X' \mapsto \phi_*(X')$ defined by the following properties:*

(a) $f_*(X') = 0$ *if the dimension of* $f(X')$ *is strictly smaller than the dimension of* X'*; and*

(b) $f_*(X') = n \cdot f(X')$ *if* $\dim f(X') = \dim X'$*, where* n *is the degree of the morphism* $X' \to f(X')$.[9]

*This assignment extends by linearity to a map $\phi_*\colon Z(X) \to Z(Y)$ of graded groups that preserves rational equivalence and hence descends to a map $\phi_*\colon A(X) \to A(Y)$ of graded groups, called the **pushforward** of Chow groups.*

Note that the definition of pushforward is subtle in at least two ways. Firstly, it is not a map of rings, only a map of graded groups. Secondly, the pushforward of the class of a subvariety is *not* necessarily the same as the class of its image. Indeed, one needs to take into account the fact that the morphism in question may have degree greater than 1; e.g., if $p \in Y$ is such that $\phi^{-1}(p)$ is the union of two (reduced) points $q, r \in X$, then $\phi_*(q) = \phi_*(r) = p$, but in order for ϕ_* to preserve rational equivalence, we require that $\phi_*(q + r) = 2p$ rather than just p.

There is an important special case of the pushforward of Chow groups, namely the pushforward along the structure morphism $X \to \operatorname{Spec} k$. It is easy to check that $A(\operatorname{Spec} k) = \mathbb{Z}$; we then obtain the following useful corollary from Proposition 42.

[9]Recall that the degree of a morphism is defined to be the degree of the extension of function fields induced by the morphism.

Corollary 43. *Let* X *be a proper scheme. Then there is a map of groups* $A(X) \to \mathbb{Z}$ *called* ***degree*** *and denoted* $\deg$ *with the following two properties:*

(a) $\deg([X']) = 0$ *if* $X' \hookrightarrow X$ *is a subvariety of positive pure dimension; and*

(b) $\deg([p]) = 1$ *if* p *is a (reduced) point.*

In particular, if $X' \hookrightarrow X$ *is a* 0*-dimensional subscheme of* X*, then* $\deg([X'])$ *is equal to the length of* X'*.*

The degree map will be very important in our enumerative applications, because we are interested in computing the lengths of the 0-dimensional subschemes that arise as degeneracy loci of sections of vector bundles.

We next introduce the pullback construction for Chow groups; the definition is a bit less subtle than the definition of pushforward, because we do not need to worry about multiplicities.

Proposition 44. *Let* $\phi\colon X \to Y$ *be a morphism of smooth quasi-projective varieties. We have the following two points:*

(a) There exists a unique map of groups $\phi^*\colon A^i(X) \to A^i(Y)$ *such that whenever* $Y' \hookrightarrow Y$ *is Cohen-Macaulay subvariety with the property that* $\dim X - \dim \phi^{-1}(Y') = \dim Y - \dim Y'$*, we have that* $\phi^*([Y']) = [\phi^{-1}(Y')]$.

(b) (Push-Pull Formula for Chow classes) Let $\alpha \in A^i(Y)$ *and* $\beta \in A_j(X)$*. Then*

$$\phi_*(\phi^*\alpha \cdot \beta) = \alpha \cdot \phi_*\beta \in A_{j-i}(Y).$$

The following corollary arises by applying the Push-Pull Formula (part (b) of Theorem 44) to the case where the map ϕ is the embedding of a closed subvariety.

Corollary 45. *With notation as in part (b) of Proposition 44, if* $\phi\colon X \hookrightarrow Y$ *is the inclusion of a closed subvariety, then for any closed subvariety* $Y' \hookrightarrow Y$*, we have*

$$[X] \cdot [Y'] = \phi_*(\phi^*[Y']).$$

Example: The Chow Ring of $\mathbb{P}^r_k$

The following proposition tells us what the Chow ring of projective space is; we shall refer to this fact many times in subsequent chapters.

Proposition 46. *Let $\zeta \in A^1(\mathbb{P}^r_k)$ denote the class of a hyperplane. For each nonnegative integer r, we have that*

$$A(\mathbb{P}^r_k) \simeq \mathbb{Z}[\zeta]/(\zeta^{r+1}).$$

The class of a codimension-m subvariety of $\mathbb{P}^r_k$ of degree d is given by $d\zeta^m$.

For example, if we take $r = 2$, then we see that the Chow ring of $\mathbb{P}^2_k$ is given by $A(\mathbb{P}^2_k) = \mathbb{Z}[\zeta]/(\zeta^3)$. The class of a plane curve of degree d is given by $d\zeta \in A^1(\mathcal{BP}^2_k)$, and the class of the intersection of a plane curve of degree d with a plane curve of degree e is given by $d\zeta \cdot e\zeta = de\zeta^2 \in A^2(\mathbb{P}^2_k)$. The degree of this class is de, which is the length of the scheme-theoretic intersection of the two curves. We have therefore re-derived the results of our rudimentary analysis in § 2.2.1.

2.3 Chern Classes

Now that we have introduced the Chow ring as a handy tool for encoding enumerative information about closed subschemes of a scheme, we need to understand the Chow classes associated to the degeneracy loci of vector bundles. To recall the setup, let $\mathcal{E}$ be a vector bundle on a scheme X, and let $\sigma_0, \dots, \sigma_i \in \Gamma(\mathcal{E})$ be global sections. From Definition 32, we know that the degeneracy locus of the sections $\sigma_0, \dots, \sigma_i$ is a closed subscheme of X, and so we can ask what its Chow class is. The following result provides us with a characterization of such Chow classes.

Theorem 47. *Let X be a smooth quasiprojective variety, and let $\mathcal{E}$ be a vector bundle on X. There exists a unique class $c(\mathcal{E}) = \sum_{i \geq 0} c_i(\mathcal{E}) \in A(X)$, called the* ***Chern class of*** *$\mathcal{E}$, with the following properties:*

(a) For each $i \geq 0$, the i^{th} ***Chern class*** *$c_i(\mathcal{E})$ is an element of the codimension-i graded component $A^i(X)$, and $c_0(\mathcal{E}) = 1$.*

(b) If $\mathcal{E} = \mathcal{L}$ is a line bundle on X, then $c_i(\mathcal{L}) = 0$ for $i \geq 2$ and $c_1(\mathcal{L})$ is the class of the divisor of zeros minus poles of any rational section of $\mathcal{L}$.

(c) If $\sigma_0, \dots, \sigma_i \in \Gamma(\mathcal{E})$ are global sections, then the Chow class of their degeneracy locus is $c_{r-i}(\mathcal{E})$. In particular, if the degeneracy locus is empty, then $c_{r-i}(\mathcal{E}) = 0$.

(d) *(Functoriality) Let* Y *be a smooth* k*-variety, and let* $\phi\colon Y \to X$ *be a morphism. Then*

$$\phi^* c(\mathcal{E}) = c(\phi^* \mathcal{E}).$$

(e) *(Whitney Formula) If we have a short exact sequence*

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_3 \longrightarrow 0$$

of vector bundles on X*, then*

$$c(\mathcal{E}_2) = c(\mathcal{E}_1) \cdot c(\mathcal{E}_3).$$

(f) *(Splitting Principle) Let* $\mathcal{E}_1, \dots, \mathcal{E}_n$ *be vector bundles on* X*, and let* $f \in A(X)$ *be a polynomial expression in* $c_i(\mathcal{E}_j)$ *with coefficients in* $\mathbb{Z}$*. If* $f = 0$ *when each* $\mathcal{E}_i$ *is isomorphic to the direct sum of line bundles, then* $f = 0$ *for all choices of the vector bundles* $\mathcal{E}_1, \dots, \mathcal{E}_n$*.*

Remark 48. We omit the proof of Theorem 47 not merely for the sake of brevity, but also because it is not immediately pertinent to our discussion of inflection points. Nonetheless, in what follows, we will explicitly prove many important corollaries of the theorem that will come in handy when we apply the tools that we have introduced in this chapter later on.

The most important takeaway from Theorem 47 is that the Chow class of the degeneracy locus of a collection of sections of a vector bundle admits a natural description that is often computationally easy to work out. Indeed, as the next lemmas demonstrate, we can combine parts (e) and (f) of the theorem, namely the Whitney Formula and the Splitting Principle, to compute Chern classes in a number of useful cases.

Lemma 49. *With notation as in Theorem 47, let* $\mathcal{E}_1, \mathcal{E}_2$ *be vector bundles on* X *such that* $\mathcal{E} \simeq \mathcal{E}_1 \oplus \mathcal{E}_2$*. Then we have*

$$c(\mathcal{E}) = c(\mathcal{E}_1) \cdot c(\mathcal{E}_2).$$

Proof. Consider the following short exact sequence:

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0$$

The lemma is an immediate consequence of applying the the Whitney Formula to the above sequence. □

Lemma 50. *With notation as in Theorem 47, let* $i = \min\{\dim X, \operatorname{rk} \mathcal{E}\}$*. Then we have that* $c_j(\mathcal{E}) = 0$ *for all* $j \geq i$.

Proof. Because $A^j(X) = 0$ for $j \geq \dim X$, we have $c_j(\mathcal{E}) = 0$ for such j. It remains to show that $c_j(\mathcal{E}) = 0$ for all $j \geq \operatorname{rk} \mathcal{E}$. Note that if $\mathcal{E}$ is itself a line bundle, then the desired result is none other than part (b) of Theorem 47. Now suppose $\mathcal{E}$ is a vector bundle with $\operatorname{rk} \mathcal{E} > 1$. If $\mathcal{E} \simeq \bigoplus_{i=1}^{\operatorname{rk} \mathcal{E}} \mathcal{L}_i$, where $\mathcal{L}_i$ is a line bundle on X for each i, then by inductively applying the result of Lemma 49, we deduce that

$$c(\mathcal{E}) = \prod_{i=1}^{\operatorname{rk} \mathcal{E}} c(\mathcal{L}_i) = \prod_{i=1}^{\operatorname{rk} \mathcal{E}} (1 + c_1(\mathcal{L}_i)).$$

Clearly, the expansion of the product on the right-hand side above has no terms of codimension larger than $\operatorname{rk} \mathcal{E}$, because each of the $\operatorname{rk} \mathcal{E}$ factors has no terms of codimension larger than 1. Thus, $c_j(\mathcal{E}) = 0$ for all $j \geq \operatorname{rk} \mathcal{E}$ when $\mathcal{E}$ is isomorphic to the direct sum of line bundles. The Splitting Principle then tells us that this result holds for all vector bundles $\mathcal{E}$. □

Lemma 51. *With notation as in Theorem 47, let* $\mathcal{L}$ *be a line bundle on* X*. Then we have*

$$c(\mathcal{E} \otimes \mathcal{L}) = \sum_{j=0}^{i} \binom{\operatorname{rk} \mathcal{E} - (i-j)}{j} \cdot c_{i-j}(\mathcal{E}) \cdot c_1(\mathcal{L})^j.$$

Proof. As a base case, suppose $\mathcal{E}$ is itself a line bundle. By part (b) of Theorem 47, if σ_1 is any rational section of $\mathcal{E}$ and σ_2 is any rational section of $\mathcal{L}$, then

$$c_1(\mathcal{E}) = \operatorname{div}_0(\sigma_1) - \operatorname{div}_\infty(\sigma_1) \quad \text{and} \quad c_1(\mathcal{L}) = \operatorname{div}_0(\sigma_2) - \operatorname{div}_\infty(\sigma_2).$$

Moreover, the tensor product $\sigma_1 \otimes \sigma_2$ is a rational section of $\mathcal{E} \otimes \mathcal{L}$, so

$$c_1(\mathcal{E} \otimes \mathcal{L}) = \operatorname{div}_0(\sigma_1 \otimes \sigma_2) - \operatorname{div}_\infty(\sigma_1 \otimes \sigma_2).$$

But by the definition of the tensor product of sections, we have that

$$\operatorname{div}_0(\sigma_1) + \operatorname{div}_0(\sigma_2) = \operatorname{div}_0(\sigma_1 \otimes \sigma_2) \quad \text{and} \quad \operatorname{div}_\infty(\sigma_1) + \operatorname{div}_\infty(\sigma_2) = \operatorname{div}_\infty(\sigma_1 \otimes \sigma_2).$$

Combining the above results, we find that

$$c_1(\mathcal{E} \otimes \mathcal{L}) = c_1(\mathcal{E}) + c_1(\mathcal{L}),$$

and this is the desired formula in the special case where $\operatorname{rk} \mathcal{E} = 1$.

Now suppose $\operatorname{rk} \mathcal{E} > 1$ and that $\mathcal{E} \simeq \bigoplus_{i=1}^{\operatorname{rk} \mathcal{E}} \mathcal{L}_i$, where $\mathcal{L}_i$ is a line bundle on X for each i. Then as in the proof of Lemma 50, we have

$$c(\mathcal{E}) = \prod_{i=1}^{\operatorname{rk} \mathcal{E}} (1 + c_1(\mathcal{L}_i)) \Longrightarrow c_i(\mathcal{E}) = s_i\big(c_1(\mathcal{L}_1), \ldots, c_1(\mathcal{L}_{\operatorname{rk} \mathcal{E}})\big),$$

where s_i denotes the i^{th} elementary symmetric polynomial for each i. Now, we also have that

$$\mathcal{E} \otimes \mathcal{L} \simeq \bigoplus_{i=1}^{\operatorname{rk} \mathcal{E}} \mathcal{L}_i \otimes \mathcal{L},$$

so by the base case and Lemma 49, we have

$$c(\mathcal{E} \otimes \mathcal{L}) = \prod_{i=1}^{\operatorname{rk} \mathcal{E}} (1 + c_1(\mathcal{L}_i) + c_1(\mathcal{L})).$$

Each term in the codimension-i component of the expansion of the product on the right-hand side above is obtained by choosing $i - j$ of the factors to be $c_1(\mathcal{L}_k)$ for some k and j of the factors to be $c_1(\mathcal{L})$. With this in mind, it is easy to check that

$$\begin{aligned} c_i(\mathcal{E} \otimes \mathcal{L}) &= \sum_{j=0}^{i} \binom{\operatorname{rk} \mathcal{E} - (i-j)}{j} \cdot s_{i-j}\big(c_1(\mathcal{L}_1), \ldots, c_1(\mathcal{L}_{\operatorname{rk} \mathcal{E}})\big) \cdot c_1(\mathcal{L})^j \\ &= \sum_{j=0}^{i} \binom{\operatorname{rk} \mathcal{E} - (i-j)}{j} \cdot c_{i-j}(\mathcal{E}) \cdot c_1(\mathcal{L})^j, \end{aligned}$$

which is the desired formula in the case where $\mathcal{E}$ is the direct sum of line bundles. Again, the Splitting Principle tells us that this result holds for all vector bundles $\mathcal{E}$. □

Remark 52. It follows either directly from the definition of the first Chern class of a line bundle $\mathcal{L}$, or from Lemma 51 applied to the fact that $\mathcal{L} \otimes \mathcal{L}^\vee \simeq \mathcal{O}_X$, that we have $c_1(\mathcal{L}) = -c_1(\mathcal{L}^\vee)$.

To conclude this section, we introduce one more concept that will appear in § 4.3

when we apply our results to studying Weierstrass points. Sometimes, we are not simply interested in degeneracy loci of *sections* of a vector bundle, but in degeneracy loci of a *map* of vector bundles. We make this notion precise as follows.

Definition 53. Let $\phi\colon \mathcal{E} \to \mathcal{F}$ be a map of vector bundles on a smooth variety X. The i^{th} **degeneracy locus** of ϕ is denoted $D_i(\phi)$ and is defined to be the closed subscheme of X that is locally given by the vanishing of the $(i+1)\times(i+1)$ minors of a matrix presentation of the map ϕ.

The following theorem expresses the Chow class of the degeneracy locus $D_i(\phi)$ in terms of the Chern classes of the bundles $\mathcal{E}$ and $\mathcal{F}$.

Theorem 54 (Porteous' Formula)**.** *Retain the setting of Definition 53, let* $e = \operatorname{rk}\mathcal{E}$, *let* $f = \operatorname{rk}\mathcal{F}$, *and let* $\gamma_i = c_i(\mathcal{E}^\vee \otimes \mathcal{F})$. *Then if* $D_i(\phi)$ *has codimension* $(\operatorname{rk}\mathcal{E} - i)(\operatorname{rk}\mathcal{F} - i)$, *its Chow class is given by*

$$[D_i(\phi)] = \begin{vmatrix} \gamma_{f-i} & \gamma_{f-i+1} & \cdots & \cdots & \gamma_{e+f-2i-1} \\ \gamma_{f-i-1} & \gamma_{f-i} & \cdots & \cdots & \gamma_{e+f-2i-2} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \gamma_{f-e+1} & \gamma_{f-e+2} & \cdots & \cdots & \gamma_{f-i} \end{vmatrix}$$

For a proof of Porteous' Formula and generalizations thereof, we refer the reader to [EH16, § 12] or [Ful98, § 14.4].

Chapter 3

Flexes, Hyperflexes, and More

> "It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out."
>
> Emil Artin, 1898–1962

The objective of the previous chapter was to introduce and develop a number of basic constructions in intersection theory — specifically, sheaves of principal parts, Chow rings, and Chern classes. In this chapter, we demonstrate how to use these tools to provide systematic solutions (in the spirit of Motivating Question 10) to enumerative problems such as Motivating Questions 11 and 18.

We begin by revisiting the problem of counting flexes on a smooth plane curve, using the Chern classes of the sheaves of principal parts to re-derive the result of Corollary 15. Subsequently, we turn to the rather more difficult problem of counting hyperflexes in families of plane curves. One might ask why hyperflexes are any more challenging to deal with than flexes. After all, the premise underlying § 2 was that computing Chern classes of the sheaves of principal parts is a more broadly applicable strategy for studying inflection points than relying on *ad hoc* constructions like the Hessian. The problem lies in the fact that even the simplest families of curves acquire singular fibers and hence fail to be smooth. Because the sheaves of principal parts fail to be locally free at singular points of the fibers of a family (see Example 31), the theory of Chern classes of vector bundles as developed in § 2.3 no longer applies, and an alternative strategy must be pursued.

Some *ad hoc* workarounds for the above problem exist in the literature; after briefly discussing the key ideas behind two of these workarounds, we present the main result of this thesis, which is essentially a new approach to studying inflection points in families of curves acquiring singular fibers.

3.1 Flexes on Plane Curves, Revisited

We now return to the matter of counting flexes on a plane curve, a problem that we solved in § 1.2.2 by introducing the Hessian, a seemingly magical construction that manages to pick out the flexes on such a curve. The question is: how do we utilize the new tools at our disposal to find a more systematic solution to this problem?

3.1.1 The Essential Computation

Recall from § 2.1.1 that we introduced the sheaves of principal parts as a bookkeeping tool that tells us when a section of a vector bundle vanishes to a specified order at a point. In the particular context of studying flexes on a plane curve C, we argued that if the list $(\sigma_1, \sigma_2, \sigma_3)$ forms a basis of $\Gamma(\mathcal{O}_{\mathbb{P}^2_k}(1))$, then the locus of flexes on C is the degeneracy locus of the triple of associated sections $\tau_{\sigma_1}, \tau_{\sigma_2}, \tau_{\sigma_3}$ of the principal parts sheaf $\mathcal{P}^3_C(\mathcal{O}_C(1))$. Moreover, if we take C to be smooth, then it follows from Corollary 30 that the sheaf $\mathcal{P}^3_C(\mathcal{O}_C(1))$ is locally free.

For now, let us make the following two assumptions:

(a) Suppose that C has only finitely many flexes, so that the locus of flexes is a 0-dimensional subscheme of C.[1]

(b) Assume further that the flex locus is reduced and that C has no hyperflexes, so that the number of flexes on C is given by the length of the flex locus.

The length of a 0-dimensional scheme is equal to the degree of its Chow class, so we need to compute the degree of the Chow class of the locus of flexes. Because the locus of flexes is the degeneracy locus of three sections of the vector bundle $\mathcal{P}^3_C(\mathcal{O}_C(1))$ that become dependent on a 0-dimensional closed subscheme of C, the Chow class of the

[1] We already know this to be true from Proposition 12, but the proof involved the Hessian, which we would like to avoid at all costs!

locus of flexes is given (by Theorem 47) to be the Chern class $c_1(\mathcal{P}^3_C(\mathcal{O}_C(1)))$. Then, the number of flexes on C, under assumptions (a) and (b) above, is simply the degree of this Chern class.

In the following proposition, we compute the desired Chern class, albeit in a considerably more general setting: our calculation holds for any smooth projective curve (rather than just a plane curve), any order of principal parts (rather than just order 3), and any line bundle on C (rather than just $\mathcal{O}_C(1)$). In § 4.2.1, we will demonstrate that this general computation also admits an elegant geometric interpretation, just as it does in the context of counting flexes on a plane curve.

Proposition 55. *Let C be a smooth projective curve of genus g, and let $\mathcal{L}$ be a line bundle on C. Then we have that*

$$c(\mathcal{P}^m_C(\mathcal{L})) = 1 + m \cdot c_1(\mathcal{L}) + \frac{m(m-1)}{2} \cdot K_C,$$

where $K_C = c_1(\Omega^1_C)$ is the ***canonical class*** *of C.*

Proof. We compute the desired Chern classes inductively, using the Whitney Formula in conjunction with the Splitting Principle (see Theorem 47) to express the Chern classes of the m^{th}-order principal parts sheaf in terms of the Chern classes of the $(m-1)^{\text{th}}$-order principal parts sheaf. Recall from Proposition 29 that we have the sequence

$$0 \longrightarrow \mathcal{L} \otimes (\operatorname{Sym}^{m-1} \Omega^1_C) \longrightarrow \mathcal{P}^m_C(\mathcal{L}) \longrightarrow \mathcal{P}^{m-1}_C(\mathcal{L}) \longrightarrow 0$$

Since we are working over a 1-dimensional space, Lemma 50 tells us that the last nonzero Chern class of any vector bundle is the first one. Using this fact and applying Lemmas 49 and 51 to the above sequence yields that

$$\begin{aligned} c(\mathcal{P}^m_C(\mathcal{L})) &= c(\mathcal{L} \otimes (\operatorname{Sym}^{m-1} \Omega^1_C)) \cdot c(\mathcal{P}^{m-1}_C(\mathcal{L})) \\ &= (1 + c_1(\mathcal{L}) + (m-1) \cdot c_1(\Omega^1_C)) \cdot (1 + c_1(\mathcal{P}^{m-1}_C(\mathcal{L}))) \\ &= 1 + c_1(\mathcal{L}) + (m-1) \cdot K_C + c_1(\mathcal{P}^{m-1}_C(\mathcal{L})). \end{aligned} \tag{3.1}$$

Since $\mathcal{P}^1_C(\mathcal{L}) = \mathcal{L}$, it follows from (3.1) by induction that

$$c(\mathcal{P}^m_C(\mathcal{L})) = 1 + c_1(\mathcal{L}) + \sum_{i=2}^{m} [c_1(\mathcal{L}) + (i-1) \cdot K_C]$$
$$= 1 + m \cdot c_1(\mathcal{L}) + \frac{m(m-1)}{2} \cdot K_C,$$

which is the desired formula. □

To apply Proposition 55 to the problem of counting flexes on a plane curve C of degree d, we take $m = 3$ and $\mathcal{L} = \iota^* \mathcal{O}_{\mathbb{P}^2_k}(1)$. We then have that

$$\deg c_1(\mathcal{P}^3_C(\mathcal{O}_C(1))) = 3 \cdot \deg c_1(\mathcal{O}_C(1)) + 3 \cdot \deg K_C.$$

To compute $\deg c_1(\mathcal{O}_C(1))$, recall that if $\iota \colon C \to \mathbb{P}^2_k$ denotes the inclusion of the curve in the plane, then $\mathcal{O}_C(1) = \iota^* \mathcal{O}_{\mathbb{P}^2_k}(1)$. Then, by [EH16, Proposition 1.31], we have

$$\iota_* c_1(\mathcal{O}_C(1)) = \iota_* c_1(\iota^* \mathcal{O}_{\mathbb{P}^2_k}(1)) = c_1(\mathcal{O}_{\mathbb{P}^2_k}(1)) \cdot [C] = \zeta \cdot d\zeta = d\zeta^2,$$

so taking degrees and using the fact that the degree is unchanged under pushforward, we find that

$$\deg c_1(\iota^* \mathcal{O}_{\mathbb{P}^2_k}(1)) = \deg \iota_* c_1(\iota^* \mathcal{O}_{\mathbb{P}^2_k}(1)) = d.$$

Also, we know from [EH16, § 1.4.1] that $\deg K_C = 2g - 2$. Combining the above results, we conclude that

$$\deg c_1(\mathcal{P}^3_C(\iota^* \mathcal{O}_{\mathbb{P}^2_k}(1))) = 3d + d(d-1)(g-1). \tag{3.2}$$

Since Corollary 15 tells us that the expected number of flexes is $3d(d-2)$, a quantity with no explicit dependence on the genus g, we must also express g in terms of d. But this is precisely what the genus-degree formula (see [EH16, Example 2.17]) does, so substituting this formula into (3.2) yields that

$$\deg c_1(\mathcal{P}^3_C(\iota^* \mathcal{O}_{\mathbb{P}^2_k}(1))) = 3d + d(d-1)\left(\binom{d-1}{2} - 1 \right) = 3d(d-2).$$

Thus, as long as the curve C satisfies the assumptions (a) and (b) stated above, we have shown that C has exactly $3d(d-2)$ flexes.

3.1.2 Verifying the Assumptions

In the next two lemmas, we check that the assumptions (a) and (b) required for the Chern class calculation of the previous section to be meaningful hold for a general plane curve C.

Lemma 56. *Let $C \subset \mathbb{P}_k^{m-1}$ be a smooth projective curve of degree $d > 1$, and let $(\sigma_1, \dots, \sigma_m)$ be a basis for $\Gamma(\mathcal{O}_{\mathbb{P}_k^{m-1}}(1))$. The degeneracy locus $V \subset C$ of the global sections $\tau_{\sigma_1}, \dots, \tau_{\sigma_m}$ is either empty or a 0-dimensional closed subscheme of C.*

Proof. Suppose the contrary, so that $\dim V = 1$. Since C is smooth and is of pure dimension 1, it is irreducible, so V is supported on all of C. However, this is impossible by the second part of the proof of [EH16, Theorem 7.13], where an analytic-local calculation is performed to show that if V is supported on all of C, then the sections $\sigma_1, \dots, \sigma_m$ fail to be linearly independent. □

Lemma 57. *For a general plane curve C of degree $d > 1$, the degeneracy locus V of the sections $\tau_{\sigma_1}, \tau_{\sigma_2}, \tau_{\sigma_3}$ of $\mathcal{P}_C^3(\mathcal{O}_C(1))$ is reduced.*

Proof. By Proposition 14, a general plane curve C has no hyperflexes. It therefore suffices to show that V is nonreduced at a point $p \in C$ precisely when p is a hyperflex of C. But this holds by the first part of the proof of [EH16, Theorem 7.13], which establishes via an analytic-local calculation that the order of vanishing of $\tau_{\sigma_1} \wedge \tau_{\sigma_2} \wedge \tau_{\sigma_3}$ is at least 2 precisely when p is a hyperflex of C. □

From Lemmas 56 and 57, we arrive at the following theorem, which gives us the count of flexes for "most" curves.

Theorem 58. *A general plane curve of degree d has exactly $3d(d-2)$ flexes (and hence has no hyperflexes).*

To top off our discussion of flexes of plane curves, it would be nice if we could provide a picture of, say, the nine flexes on a elliptic curve in the plane. As strange as this may seem, it is in fact impossible to do so; we prove this claim as follows.

Proposition 59. *Let $k = \mathbb{C}$, and let $E \subset \mathbb{P}_{\mathbb{C}}^2$ be an elliptic curve. Not all flexes of C are defined over $\mathbb{R}$. In particular, it is impossible to draw a real picture of the flexes on a plane curve.*

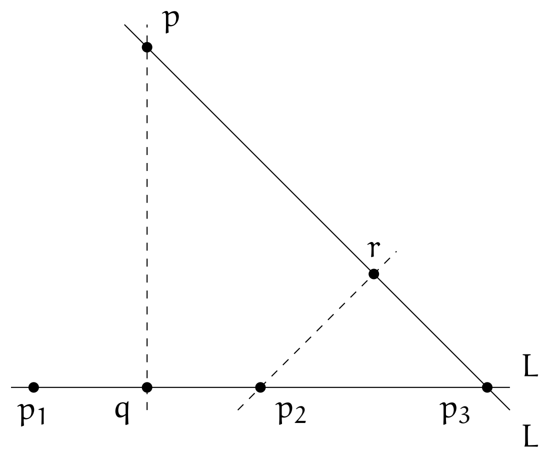

Figure 3.1: The relevant constructions in the proof of Theorem 60.

Proof. If a, b are flexes of E, then a, b are 3-torsion points of E (see Example 16). But then $a + b$ is evidently also a 3-torsion point, and hence a flex, of E. Therefore, the nine flexes of E satisfy the following property: every line joining two flexes of E necessarily passes through a third flex of E. The following result, known as the Sylvester–Gallai Theorem,[2] asserts that it is impossible for any finite set of points in the real plane to have this property.

Theorem 60 (Sylvester–Gallai)**.** *Let $S \subset \mathbb{R}^2$ be any finite collection of points, not all of which are collinear. There exists a line $L \subset \mathbb{R}^2$ passing through exactly two elements of S.*

Proof. While reading this proof, due to L. Kelly, it may be helpful to refer to Figure 3.1. Choose a line $L \subset \mathbb{R}^2$ passing through at least two points of S and a point $p \in S \setminus L$ so that the (perpendicular) distance between p and L is minimal over all such choices of line and point. Suppose for the sake of contradiction that L passes through at least three points of S, call them p_1, p_2, p_3. Let the perpendicular line to L through p meet L at q. We can think of q as splitting the line L into two sides; assume without loss of generality that p_2, p_3 lie on the same side and that p_2 is closer to q than p_3. Let L' be the line passing through p and p_3, and let the perpendicular line to L' through p_2 meet L' at r. Then the triangle p_3qp is similar to the triangle p_3rp_2, so because the distance between p_2 and p_3 is smaller than the distance between q and p_3, which is in turn smaller than the distance between p and p_3, it follows that the distance between p_2 and L' is smaller than the distance between p and L, which is a contradiction. □

[2]Historical aside: This question was first posed by J. J. Sylvester in 1893 and was thereafter independently proven by E. Melchior in 1941 and T. Gallai in 1944.

The proposition follows from Theorem 60 upon observing that the elliptic curve E cannot contain 9 collinear points. □

3.2 A First Look at Counting Hyperflexes

Enough about flexes for now — let us turn our attention to the case of hyperflexes. As described in § 1.3, a general plane curve of degree $d > 3$ fails to have any hyperflexes; on the other hand, the locus of plane curves of degree d with hyperflexes is a hypersurface in the parameter space $\mathbb{P}_k^{N(d)}$ of degree-d plane curves, so if we look at a **1-parameter family of curves** (i.e., the curves associated to a 1-dimensional subvariety of $\mathbb{P}_k^{N(d)}$), we expect a finite number of them to have hyperflexes. For much of the rest of this thesis, we shall concern ourselves with the study of hyperflexes, as well as other kinds of inflection points, on the members of such 1-parameter families of curves.

3.2.1 The Main Issue: Singular Fibers

As we observed at the beginning of this chapter, the main quagmire that arises in the context of counting hyperflexes is that families of curves tend to have singular members. To see why this is the case, we consider the "simplest" example of a 1-parameter family — namely, a **pencil** of hypersurfaces in projective space.

Let d, r be positive integers. Recall that a pencil of hypersurfaces of degree d in $\mathbb{P}_k^r$ is, roughly speaking, a family of hypersurfaces corresponding to the points of a line in the projective space parameterizing all such hypersurfaces. In more concrete terms, every pencil can be constructed as follows. Take two distinct homogeneous polynomials $F, G \in k[X_0, \dots, X_r]$ of degree d, and consider the vanishing locus X in $\mathbb{P}_k^r \times_k \mathbb{P}_k^1$ of the polynomial

$$s \cdot F(X_0, \dots, X_r) + t \cdot G(X_0, \dots, X_r),$$

where we take projective coordinates $[X_0 : \cdots : X_r]$ on the $\mathbb{P}_k^r$ factor and $[s : t]$ on the $\mathbb{P}_k^1$ factor. Then X, viewed as the total space of a family over $\mathbb{P}_k^1$ via the natural projection map, is what we call a pencil of hypersurfaces of degree d. In what follows, we shall provide an explicit count of the number of singular members that occur in a general such pencil.

Given a pencil of hypersurfaces of degree d in $\mathbb{P}^r_k$, we ask the following question: how can we describe the locus of points $p \in \mathbb{P}^r_k$ such that some element of the pencil is singular at p? In the spirit of § 2.1.1, we can think of hypersurfaces of degree d in $\mathbb{P}^r_k$ as vanishing loci of global sections of the line bundle $\mathcal{O}_{\mathbb{P}^r_k}(d)$. From this perspective, our question can be restated as follows: given two distinct global sections $F, G \in \Gamma(\mathcal{O}_{\mathbb{P}^r_k}(d))$, how can we describe the locus of points $p \in \mathbb{P}^r_k$ for which there exist $a, b \in k$ such that the section $a \cdot F + b \cdot G \in \Gamma(\mathcal{O}_{\mathbb{P}^r_k}(d))$ has order of vanishing at least 2 at p (so that $a \cdot F + b \cdot G$ is singular at p), in the sense that the image of $a \cdot F + b \cdot G$ under the map

$$\Gamma(\mathcal{O}_{\mathbb{P}^r_k}(d)) \to \Gamma(\mathcal{O}_{\mathbb{P}^r_k}(d) \otimes \mathcal{O}_{\mathbb{P}^r_k}/\mathcal{I}_p^2)$$

is the zero-section? It follows that the desired locus is the degeneracy locus of the sections $\tau_F, \tau_G \in \Gamma(\mathcal{P}^2_{\mathbb{P}^r_k}(\mathcal{O}_{\mathbb{P}^r_k}(d)))$. Suppose that the following conditions hold:

(a) The degeneracy locus of τ_F and τ_G is reduced; and

(b) τ_F and τ_G degenerate in the expected codimension, so that only finitely many of the members of the pencil are singular.

Then the number of singular members of the pencil is given by degree of the Chow class of the degeneracy locus, which by Theorem 47 is simply $\deg c_r(\mathcal{P}^2_{\mathbb{P}^r_k}(\mathcal{O}_{\mathbb{P}^r_k}(d)))$. In the following lemma, we compute this Chern class in somewhat greater generality.

Lemma 61. *Let $\mathcal{L}$ be a line bundle on $\mathbb{P}^r_k$. Then we have that*

$$c(\mathcal{P}^2_{\mathbb{P}^r_k}(\mathcal{L})) = (1 + c_1(\mathcal{L}) - \zeta)^{r+1}.$$

Proof. Recall that we have the following short exact sequence, known as the Euler sequence (for a proof, see [Vak17, Theorem 21.4.6])

$$0 \longrightarrow \Omega^1_{\mathbb{P}^r_k} \longrightarrow (\mathcal{O}_{\mathbb{P}^r_k}(-1))^{\oplus(r+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^r_k} \longrightarrow 0$$

Tensoring with $\mathcal{L}$ yields that

$$0 \longrightarrow \mathcal{L} \otimes \Omega^1_{\mathbb{P}^r_k} \longrightarrow (\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^r_k}(-1))^{\oplus(r+1)} \longrightarrow \mathcal{L} \longrightarrow 0$$

But notice by Proposition 29 that the sheaf of principal parts $\mathcal{P}^2_{\mathbb{P}^r_k}(\mathcal{L})$ fits in as the middle term of the same short exact sequence:

$$0 \longrightarrow \mathcal{L} \otimes \Omega^1_{\mathbb{P}^r_k} \longrightarrow \mathcal{P}^2_{\mathbb{P}^r_k}(\mathcal{L}) \longrightarrow \mathcal{L} \longrightarrow 0$$

Then, by the Whitney Formula (see Theorem 47), it follows that

$$c(\mathcal{P}^2_{\mathbb{P}^r_k}(\mathcal{L})) = c((\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^r_k}(-1))^{\oplus(r+1)}).$$

The fact that $\mathcal{O}_{\mathbb{P}^r_k}(-1) \otimes \mathcal{O}_{\mathbb{P}^r_k}(1) \simeq \mathcal{O}_{\mathbb{P}^r_k}$, together with Lemma 51, implies that

$$c(\mathcal{O}_{\mathbb{P}^r_k}(-1)) = 1 - \zeta.$$

We then have by Lemmas 49 and 51 that

$$c((\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^r_k}(-1))^{\oplus(r+1)}) = c(\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^r_k}(-1))^{r+1} = (1 + c_1(\mathcal{L}) - \zeta)^{r+1},$$

which is the desired result. □

In the situation of enumerating the singular members of a pencil of degree-d hypersurfaces in $\mathbb{P}^r_k$, we take $\mathcal{L} = \mathcal{O}_{\mathbb{P}^r_k}(d)$, so that $c_1(\mathcal{L}) = d\zeta$. Substituting this into the result of Lemma 61, we find that

$$\deg c_r(\mathcal{P}^2_{\mathbb{P}^r_k}(\mathcal{O}_{\mathbb{P}^r_k}(d))) = \deg((r+1)(d-1)^r \zeta^r) = (r+1)(d-1)^r.$$

It remains to verify the assumptions (a) and (b) stated immediately before Lemma 61. It turns out that both hold if the pencil is chosen generally among all pencils in the parameter space $\mathbb{P}^{N(d)}_k$; see [EH16, § 7.3.1] for a proof of assumption (a) and [EH16, Proposition 7.1] for a proof of assumption (b). At the end of the day, we arrive at the following useful result.

Corollary 62. *A general pencil of hypersurfaces of degree* d *in* $\mathbb{P}^r_k$ *has exactly* $(r+1)(d-1)^r$ *singular members. In particular a general pencil of plane curves of degree* d *has* $3(d-1)^2$ *singular members.*

Remark 63. The singular hypersurfaces of degree d in $\mathbb{P}^r_k$ form a hypersurface, called the **discriminant**, in the projective space parameterizing all degree-d hypersurfaces. Corollary 62 can be rephrased to say that the degree of the discriminant hypersurface is $(r+1)(d-1)^r$.

3.2.2 Hyperflexes in a Pencil: What's Known?

We have shown that any pencil of plane curves of degree d contains singular elements. How, then, can we compute the number of hyperflexes on such a pencil? Two workarounds for this obstacle exist in the literature, but they have the disadvantage of being *ad hoc*, and hence limited in scope. Before we introduce our own approach to dealing with singular elements, we briefly discuss these two existing strategies.

Strategy I: Look at Hyperflex Point-Line Pairs

The first method, which is detailed in [EH16, § 11.3], works not by enumerating the *points* that occur as hyperflexes, but by enumerating the hyperflex *point-line pairs*. To see how this works, recall from the proof of Proposition 14 that we introduced the point-line incidence correspondence

$$Z = \{(\text{point } p, \text{line } L) \in \mathbb{P}_k^2 \times \mathbb{P}_k^{N(1)} : p \in L\}.$$

Let $\pi_1 : Z \to \mathbb{P}_k^2$ be the projection map onto the "point" factor, and consider the sheaf

$$\mathcal{E} = \mathcal{P}^4_{Z/\mathbb{P}_k^{N(1)}}\left(\pi_1^* \mathcal{O}_{\mathbb{P}_k^2}(d)\right),$$

where we view Z as a family over $\mathbb{P}_k^{N(1)}$ by the projection map onto the "line" factor. Note that $\mathcal{E}$ is locally free by Corollary 30 because the map $Z \to \mathbb{P}_k^{N(1)}$ is smooth. Moreover, it follows from Proposition 27 that the fiber of $\mathcal{E}$ at a pair $(p, L) \in Z$ is

$$\mathcal{E}|_{(p,L)} = \Gamma(\mathcal{O}_L(d) \otimes \mathcal{O}_L/\mathcal{I}_p^4),$$

so since $\dim_k \Gamma(\mathcal{O}_L(d) \otimes \mathcal{O}_L/\mathcal{I}_p^4) = 4$, we have that $\mathcal{E}$ is a vector bundle on Z of rank 4. Roughly speaking, if F, G are the homogeneous polynomials defining our pencil, then F, G give rise to global sections σ_F, σ_G of $\mathcal{E}$, and it is not hard to check that the locus in Z of point-line pairs (p, L) that arise as hyperflexes on elements of our pencil is the degeneracy locus of the sections σ_F and σ_G. Since the locus of plane curves of degree d that have hyperflexes forms a hypersurface in $\mathbb{P}_k^{N(d)}$, a general pencil of plane curves has only finitely elements with a hyperflex. It then follows from Theorem 47 that the number of hyperflexes on a general pencil of plane curves is given by $\deg c_3(\mathcal{E})$. To compute this Chern class requires one to determine the Chow ring of Z, which would further require us to introduce the notion of **projective bundle**; see [EH16, § 9] for a thorough discuss of projective bundles and their Chow rings. At any rate, it is possible to show that $\deg c_3(\mathcal{E}) = 6(d-3)(3d-2)$, so we end up with the following theorem.

Theorem 64. *A general pencil of plane curves of degree* $d \geq 3$ *has exactly* $6(d-3)(3d-2)$ *hyperflexes (and no higher order flexes).*

Strategy II: The "Hilbert Scheme of Nodal Curves"

The second method, which was developed by Z. Ran in [Ran13], only works when the singular fibers of the family are nodal, because it relies on specific properties of what Ran terms "**the Hilbert scheme of nodal curves**," by which he actually means the punctual flag Hilbert scheme parameterizing schemes of bounded length supported at individual points of the fibers of the family. To describe Ran's work (and subsequently, our new results), we need to refine our notion of a family of curves from Definition 24; we do this as follows.

Definition 65. Let X/B be a family as in Definition 24. Suppose that B is 1-dimensional. We say that the family X/B is **admissible** if the following conditions hold:

(a) Each (geometric) fiber is a local complete intersection curve;

(b) Each fiber is **Gorenstein**, so that the **relative dualizing sheaf** $\omega_{X/B}$ is invertible;

(c) Only finitely many of the fibers are singular; and

(d) Each singular fiber contains exactly one planar singularity.

For an admissible family X/B, we denote by $\Gamma \subset X$ the locus of points where the map π fails to be smooth (namely, the singular points of the singular fibers) and by $U = X \setminus \Gamma$ the complement.

We are now in position to describe Ran's result. Let X/B be an admissible family with the property that each singular fiber is nodal, and let $\mathcal{E}$ be a vector bundle on X. In this general setting, Ran introduces a "tautological bundle" $\Lambda_m(\mathcal{E})$ defined as follows. Let $X_B^{[m]} = \mathrm{Hilb}_m(X/B)$ denote the relative Hilbert scheme parameterizing length-m subschemes of the fibers of the map $\pi\colon X \to B$, and let π_1, π_2 be the projection maps from $X_B^{[m]} \times_B X$ onto the left and right factors, respectively. Then we take

$$\Lambda_m(\mathcal{E}) = \pi_{1*}\big(\pi_2^*\mathcal{E} \otimes \mathcal{O}_{X_B^{[m]} \times_B X}/\mathcal{I}_m\big),$$

where $\mathcal{I}_m$ is the universal ideal sheaf of colength m in $\mathcal{O}_{X_B^{[m]} \times_B X}$. Since we are not interested in all length-m subschemes of the fibers, but only in those subschemes that are supported at a single point, we ought to consider the pullback of the tautological bundle $\Lambda_m(\mathcal{E})$ to the punctual Hilbert scheme, which Ran denotes $\Gamma_{(m)}$, parameterizing length-m schemes supported at individual points of the fibers. It turns out that the Chow class of the locus of points in X that are hyperflexes for their corresponding fibers is given by

$$c_2(\Lambda_m(\mathcal{O}_X(1))|_{\Gamma_{(m)}}).$$

The computation of the above Chern class is rather involved, a key reason being that the punctual Hilbert scheme $\Gamma_{(m)}$ is generally singular. Consequently, Ran ends up working not over $\Gamma_{(m)}$ itself but over the aforementioned punctual flag Hilbert scheme, which turns out to be an iterated blowup of $\Gamma_{(m)}$. After much heavy lifting, Ran arrives at the following elegant result.

Theorem 66 ([Ran13, Example 3.21])**.** *Let X/B be an admissible family with each singular fiber nodal, let $\mathcal{L}$ be a line bundle on X, and let the number of singular fibers of X/B be denoted by δ. Then we have*

$$c_2(\Lambda_m(\mathcal{L})|_{\Gamma_{(m)}}) = \binom{m}{2} \cdot c_1(\mathcal{L})^2 + \left(3\binom{m+1}{4} - \binom{m}{3}\right) \cdot c_1(\omega_{X/B})^2 + \left(3\binom{m+1}{3} - 2\binom{m}{2}\right) \cdot c_1(\omega_{X/B}) \cdot c_1(\mathcal{L}) - \binom{m+1}{4} \cdot \delta.$$

Remark 67. We make the following observations:

(a) By taking the family X/B to be a pencil of plane curves of degree d and by taking $\mathcal{L} = \mathcal{O}_X(1)$, one can show that the formula for the number of hyperflexes in Theorem 64 is a corollary of Theorem 66. For a proof of this assertion, refer to § 3.4.2, where we apply our new results to the case of counting hyperflexes in a pencil.

(b) It may be possible to generalize Ran's strategy to families of curves acquiring higher-order singularities. Indeed, based on ideas introduced by Ran in [Ran05c], H. Lee has found a description of the punctual Hilbert scheme of length-m schemes supported at a cusp [Lee12]. It would be certainly be quite interesting if analogues of the tautological module on families of nodal curves and the consequent enumerative formula can be derived for families of cuspidal curves using Lee's results. In any case, the strategy that we use to re-derive Theorem 66 can be extended to handle families acquiring higher-order singularities; see § 3.4.3 for a more detailed discussion of this subject.

(c) For more references on how one can use the "Hilbert scheme of nodal curves" to solve interesting enumerative problems on such curves, see the articles [Ran05b] and [Ran05a].

3.3 A New Approach

In this section, we present a new method of circumventing the problem posed by the presence of singular fibers in studying inflection points on families of curves. Our strategy comprises three key steps:

(a) Obtain locally free replacements for the sheaves of principal parts;

(b) Compute the Chern classes of these replacement sheaves; and

(c) Subtract any contributions to these Chern classes that arise from singular points (because we do not regard such points as inflection points; see Definition 6).

3.3.1 Replacing the Sheaves of Principal Parts

As described in § 3.2.1, the key issue is that the sheaves of principal parts fail to be locally free at the singular points of the fibers the family X/B. Since we do not consider singular points of curves to be inflection points anyway, one might wonder whether it is possible to replace the sheaves of principal parts with new sheaves that have the following two properties:

(a) They must be locally free on all of X, so that we can make sense of and compute their Chern classes; and

(b) They must be isomorphic to the sheaves of principal parts on the complement of the singular points of the fibers, so that they serve the same purpose.

In what follows, we shall answer this question in the affirmative.

Defining the Replacement Sheaves

As the following result indicates, the desired replacement sheaves are none other than double-duals of the sheaves of principal parts.

Theorem 68. *Let X/B be an admissible family, and let $\mathcal{E}$ be a vector bundle on X. The double-dual sheaf $\mathcal{P}^m_{X/B}(\mathcal{E})^{\vee\vee}$ is the unique locally free sheaf on X whose restriction to U is isomorphic to $\mathcal{P}^m_{X/B}(\mathcal{E})|_U$.*

For the sake of convenience, we give the double-duals of the sheaves of principal parts a special name.

Definition 69. With notation as in Theorem 68, we say that $\mathcal{P}^m_{X/B}(\mathcal{E})^{\vee\vee}$ is the m^{th}**-order sheaf of invincible parts** associated to the family X/B.

To prove Theorem 68, we first recall three important facts about coherent sheaves; we omit proofs of these facts because they are well-known.

Proposition 70. *Let S be a Noetherian integral scheme. We have the following properties pertaining to coherent sheaves on S:*

(a) Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_S$-modules. Then the dual sheaf $\mathcal{F}^\vee$ is reflexive (i.e., isomorphic to its own double-dual).

(b) Suppose S is normal, and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be reflexive sheaves on S with the property that they differ on a locus of codimension at least 2. Then $\mathcal{F}_1 \simeq \mathcal{F}_2$.

(c) Suppose S is smooth, and let $\mathcal{F}$ be a coherent sheaf on S. If $\mathcal{F}$ is reflexive, then it is locally free away from a locus of codimension at least 3.

Proof. The above properties are well-known; a good reference for the basic facts about reflexive sheaves is [Har80], in which property (a) is Corollary 1.2, property (b) is Proposition 1.6, and property (c) is Corollary 1.4. □

We are now in position to provide a proof of Theorem 68.

Proof of Theorem 68. Because the sheaf of principal parts $\mathcal{P}^m_{X/B}(\mathcal{E})$ is coherent (for instance, it is the pushforward of the coherent sheaf $\pi_2^*\mathcal{E} \otimes \mathcal{O}_{X\times_B X}/\mathcal{I}_\Delta^m$ along the proper morphism π_1 to the Noetherian scheme B and must therefore be coherent) and because

X was taken to be Noetherian and smooth, the assumptions underlying each property in the statement of Proposition 70 are satisfied by taking $S = X$ and $\mathcal{F} = \mathcal{P}^m_{X/B}(\mathcal{E})$. By part (a) of Proposition 70, the dual sheaf $\mathcal{P}^m_{X/B}(\mathcal{E})^\vee$ is reflexive; then, part (c) of the proposition tells us that $\mathcal{P}^m_{X/B}(\mathcal{E})^\vee$ is locally free because $\dim X = 2 < 3$. It follows that double-dual sheaf $\mathcal{P}^m_{X/B}(\mathcal{E})^{\vee\vee}$ is also locally free, and part (b) of the proposition guarantees uniqueness. □

Chern Classes of Invincible Parts

If the sheaves of invincible parts are going to be of any help in our quest to enumerate inflection points, then we shall need a way to compute their Chern classes. Recall from Proposition 55 that we computed the Chern classes of the sheaves of principle parts by inductively using the fact that they fit into exact sequences (see Corollary 30). The next proposition shows that an analogous result holds for the sheaves of invincible parts.

Proposition 71. *Recall the setting of Theorem 68. For each integer $m \geq 2$, we have the following short exact sequence:*

$$0 \longrightarrow \mathcal{E} \otimes \omega_{X/B}^{\otimes(m-1)} \longrightarrow \mathcal{P}^m_{X/B}(\mathcal{E})^{\vee\vee} \longrightarrow \mathcal{P}^{m-1}_{X/B}(\mathcal{E})^{\vee\vee} \longrightarrow 0$$

Proof. Let $\mathcal{K}^m(\mathcal{E})$ denote the kernel of the surjective map $\mathcal{P}^m_{X/B}(\mathcal{E}) \twoheadrightarrow \mathcal{P}^{m-1}_{X/B}(\mathcal{E})$. Then $\mathcal{K}^m(\mathcal{E})$ is a coherent sheaf, and so by parts (a) and (c) of Proposition 70, its dual is reflexive and hence locally free. But we have the following identifications of sheaves restricted to the open subscheme U:

$$(\mathcal{K}^m(\mathcal{E})^\vee)|_U \simeq (\mathcal{E} \otimes \operatorname{Sym}^{m-1} \Omega^1_{X/B})|_U \simeq (\mathcal{E} \otimes \omega_{X/B}^{\otimes(m-1)})|_U.$$

It then follows from part (b) of Proposition 70 and the fact that $\omega_{X/B}$ is locally free that $\mathcal{K}^m(\mathcal{E})^\vee \simeq \mathcal{E} \otimes \omega_{X/B}^{\otimes(m-1)}$. Thus, taking the dual of the short exact sequence established in Proposition 29, we obtain the following exact sequence:

$$0 \longrightarrow \mathcal{P}^{m-1}_{X/B}(\mathcal{E})^\vee \longrightarrow \mathcal{P}^{m}_{X/B}(\mathcal{E})^\vee \longrightarrow \mathcal{E} \otimes \omega_{X/B}^{\otimes(m-1)} \longrightarrow$$
$$\mathcal{E}xt^1(\mathcal{P}^{m-1}_{X/B}(\mathcal{E}), \mathcal{O}_X) \to \mathcal{E}xt^1(\mathcal{P}^{m}_{X/B}(\mathcal{E}), \mathcal{O}_X) \to \mathcal{E}xt^1(\mathcal{E} \otimes \omega_{X/B}^{\otimes(m-1)}, \mathcal{O}_X) \longrightarrow$$
$$\mathcal{E}xt^2(\mathcal{P}^{m-1}_{X/B}(\mathcal{E}), \mathcal{O}_X) \to \mathcal{E}xt^2(\mathcal{P}^{m}_{X/B}(\mathcal{E}), \mathcal{O}_X) \to \mathcal{E}xt^2(\mathcal{E} \otimes \omega_{X/B}^{\otimes(m-1)}, \mathcal{O}_X) \to 0$$

where we have used the fact that the sheaves $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X)$ are equal to 0 for each $i > 2$ and $\mathcal{F} \in S := \{\mathcal{P}^n_{X/B}(\mathcal{E})^\vee, \mathcal{E} \otimes \omega_{X/B}^{\otimes n} : n \geq 1\}$ because $\dim X = 2$. But we know that each $\mathcal{F} \in S$ is locally free away from the 0-dimensional subscheme Γ, so $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X)$ is the direct sum of finitely many skyscraper sheaves for each $i \in \{1, 2\}$ and $\mathcal{F} \in S$. Either by applying parts (a) and (c) of Proposition 70 or by appealing to the easy fact that the dual of a skyscraper sheaf is the zero-sheaf, we deduce that $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X)^\vee = 0$ for each $i \in \{1, 2\}$ and $\mathcal{F} \in S$. Taking duals of the above dual exact sequence, we obtain the short exact sequence in the statement of the proposition. □

We now use Proposition 71 to compute the Chern classes of the sheaves of invincible parts of a line bundle.

Proposition 72. *Let $\mathcal{L}$ be a line bundle on X, and let the number of singular fibers of the family X/B be denoted by δ. Then we have*

$$\begin{aligned} c(\mathcal{P}^m_{X/B}(\mathcal{L})^{\vee\vee}) = 1 + m \cdot c_1(\mathcal{L}) + \binom{m}{2} \cdot c_1(\omega_{X/B}) + \\ \binom{m}{2} \cdot c_1(\mathcal{L})^2 + \left(3\binom{m+1}{4} - \binom{m}{3}\right) \cdot c_1(\omega_{X/B})^2 + \\ \left(3\binom{m+1}{3} - 2\binom{m}{2}\right) \cdot c_1(\omega_{X/B}) \cdot c_1(\mathcal{L}). \end{aligned}$$

Proof. The proof proceeds in a fashion analogous to that of Proposition 55. By Proposition 71, we have the short exact sequence

$$0 \longrightarrow \mathcal{L} \otimes \omega_{X/B}^{\otimes(m-1)} \longrightarrow \mathcal{P}^m_{X/B}(\mathcal{L})^{\vee\vee} \longrightarrow \mathcal{P}^{m-1}_{X/B}(\mathcal{L})^{\vee\vee} \longrightarrow 0$$

Since we are working over the 2-dimensional space X, the last nonzero Chern class of any vector bundle is the second one. Using this fact and applying the Whitney Formula in conjunction with the Splitting Principle (see Theorem 47) to the above sequence yields the following:

$$
\begin{aligned}
&c(\mathcal{P}^m_{X/B}(\mathcal{L})^{\vee\vee}) = \\
&c(\mathcal{L} \otimes \omega_{X/B}^{\otimes(m-1)})) \cdot c(\mathcal{P}^{m-1}_{X/B}(\mathcal{L})^{\vee\vee}) = \\
&(1 + c_1(\mathcal{L}) + (m-1)\cdot c_1(\omega_{X/B}))\cdot(1 + c_1(\mathcal{P}^{m-1}_{X/B}(\mathcal{L})^{\vee\vee}) + c_2(\mathcal{P}^{m-1}_{X/B}(\mathcal{L})^{\vee\vee})) = \\
&1 + c_1(\mathcal{L}) + (m-1)\cdot c_1(\omega_{X/B}) + c_1(\mathcal{P}^{m-1}_{X/B}(\mathcal{L})^{\vee\vee}) + \\
&\big(c_1(\mathcal{L}) + (m-1)\cdot c_1(\omega_{X/B})\big)\cdot c_1(\mathcal{P}^{m-1}_{X/B}(\mathcal{L})^{\vee\vee}) + c_2(\mathcal{P}^{m-1}_{X/B}(\mathcal{L})^{\vee\vee})
\end{aligned} \tag{3.3}
$$

As it happens, the calculation of the first Chern class performed in the proof of Proposition 55 applies to the present situation with the canonical class K_C replaced by $c_1(\omega_{X/B})$, so we have that

$$
c_1(\mathcal{P}^m_{X/B}(\mathcal{L})^{\vee\vee}) = 1 + m\cdot c_1(\mathcal{L}) + \binom{m}{2}\cdot c_1(\omega_{X/B}). \tag{3.4}
$$

As for the second Chern class, substituting the result of (3.4) into the terms of codimension-2 in (3.3) and applying induction yields that

$$
\begin{aligned}
c_2(\mathcal{P}^m_{X/B}(\mathcal{L})^{\vee\vee}) = \sum_{i=2}^{m}\Big[(i-1)\cdot c_1(\mathcal{L})^2 + \Big(\tbinom{i-1}{2} + (i-1)^2\Big)\cdot c_1(\omega_{X/B})\cdot c_1(\mathcal{L}) + \\
(i-1)\cdot\tbinom{i-1}{2}\cdot c_1(\omega_{X/B})^2\Big],
\end{aligned}
$$

and evaluating the above sum using the standard identities for summing consecutive squares and cubes yields the desired formula. □

Wronski Algebra Systems

The idea of finding locally free replacements for the sheaves of principal parts over singular curves dates back to the work of D. Laksov and A. Thorup from two decades ago. In [LT94], they introduce the notion of a Wronski algebra system, which is motivated and defined as follows.

Let X/B be an admissible family, and let $\mathcal{E}$ be a vector bundle on X. In some sense, the failure of $\Omega^1_{X/B}$ to be locally free is the reason why the sheaves of principal parts $\mathcal{P}^m_{X/B}(\mathcal{E})$ fail to be locally free; indeed, in the proof of Corollary 30, we used the local-freeness of $\Omega_{X/B}$ in the case where the family X/B is smooth to inductively deduce the local-freeness of $\mathcal{P}^m_{X/B}(\mathcal{E})$. But because the relative dualizing sheaf $\omega_{X/B}$ is the unique locally free replacement for the sheaf of relative differentials $\Omega^1_{X/B}$, it is natural to ask whether we can come up with a sequence of sheaves $\mathcal{Q}^m_{X/B}(\mathcal{E})$ that satisfy the same basic properties as the sheaves of principal parts, but fit into exact sequences having tensor powers of the relative dualizing sheaf as the kernel. To this end, we have the following definition:

Definition 73. A **Wronski algebra system** associated to the pair $(X/B, \mathcal{E})$ is a sequence of sheaves $\mathcal{Q}^m_{X/B}(\mathcal{E})$ satisfying the following properties:

(a) For each $m \geq 1$ we have maps $\psi_m \colon \mathcal{P}^m_{X/B}(\mathcal{E}) \to \mathcal{Q}^m_{X/B}(\mathcal{E})$ such that the following diagram commutes, with each row being a short exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}^m(\mathcal{E}) & \longrightarrow & \mathcal{P}^m_{X/B}(\mathcal{E}) & \longrightarrow & \mathcal{P}^{m-1}_{X/B}(\mathcal{E}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \psi_m & & \downarrow \psi_{m-1} & & \\ 0 & \longrightarrow & \mathcal{E} \otimes \omega^{\otimes(m-1)}_{X/B} & \longrightarrow & \mathcal{Q}^m_{X/B}(\mathcal{E}) & \longrightarrow & \mathcal{Q}^{m-1}_{X/B}(\mathcal{E}) & \longrightarrow & 0 \end{array}$$

(b) The map $\psi_1 \colon \mathcal{E} = \mathcal{P}^1_{X/B}(\mathcal{E}) \to \mathcal{Q}^1_{X/B}(\mathcal{E})$ is an isomorphism.

Perhaps the most interesting results on Wronski algebra systems are due to E. Esteves, who proved the following theorem.

Theorem 74 ([Est96, Theorem 2.6])**.** *Let X/B be an admissible family with $\dim B$ arbitrary, and let $\mathcal{E}$ be a vector bundle on X. There exists a unique Wronski algebra system associated to the pair $(X/B, \mathcal{E})$.*

When the base B is 1-dimensional, Theorem 68 in conjunction with Proposition 71 demonstrate that the unique Wronski algebra system guaranteed by Theorem 74 is none other than the sequence of sheaves of invincible parts; i.e., we have $\mathcal{Q}^m_{X/B}(\mathcal{E}) = \mathcal{P}^m_{X/B}(\mathcal{E})^{\vee\vee}$ and $\psi_m = \mathrm{can}_{\mathrm{ev}}$, where

$$\mathrm{can}_{\mathrm{ev}} \colon \mathcal{P}^m_{X/B}(\mathcal{E}) \to \mathcal{P}^m_{X/B}(\mathcal{E})^{\vee\vee}$$

denotes the canonical map from a sheaf to its double dual.[3]

[3] That $\psi_m = \mathrm{can}_{\mathrm{ev}}$ follows from the fact that the canonical map may be regarded as a natural transformation between the identity and double-dual functors on the category of sheaves of $\mathcal{O}_X$-modules.

3.3.2 Dealing with the Singular Points

How can we use the sheaves of invincible parts to count, say, hyperflexes in a pencil X/B of plane curves of degree d? As we did in the case of counting flexes, take a basis $(\sigma_1, \sigma_2, \sigma_3)$ of $\Gamma(\mathcal{O}_{\mathbb{P}^2_k}(1))$, and let $p \in X$ be a smooth point of its corresponding fiber $X_{\pi(p)}$. The condition that some line in the plane is a hyperflex line for $X_{\pi(p)}$ at p is equivalent to the condition that there exist some scalars $a_1, a_2, a_3 \in k$ such that

$$\text{mult}_p(C, V(\sigma)) \geq 4, \tag{3.5}$$

where $\sigma = a_1 \cdot \sigma_1 + a_2 \cdot \sigma_2 + a_3 \cdot \sigma_3$. By Lemma 23, this is equivalent to the condition that σ vanish to order 4 at p, which is the same as saying that the vanishing locus of $\tau_\sigma \in \Gamma(\mathcal{P}^4_{X/B}(\mathcal{O}_X(1)))$ contains p. So where do the sheaves of invincible parts come in? Let ξ_σ denote the image of τ_σ under the map of global sections induced by the canonical map of sheaves

$$\text{can}_{\text{ev}} \colon \mathcal{P}^4_{X/B}(\mathcal{O}_X(1)) \to \mathcal{P}^4_{X/B}(\mathcal{O}_X(1))^{\vee\vee}.$$

Since the sheaves of invincible parts are isomorphic via can_{ev} to the sheaves of principal parts on a small-enough open neighborhood of p (namely, an open neighborhood containing no points at which the corresponding fiber of the family is singular), the condition (3.5) is equivalent to the condition that the vanishing locus of $\xi_\sigma \in \Gamma(\mathcal{P}^4_{X/B}(\mathcal{O}_X(1))^{\vee\vee})$ contains p.

Since $\xi_\sigma = a_1 \cdot \xi_{\sigma_1} + a_2 \cdot \xi_{\sigma_2} + a_3 \cdot \xi_{\sigma_3}$, it follows that the locus of hyperflexes is the degeneracy locus of the sections $\xi_{\sigma_1}, \xi_{\sigma_2}, \xi_{\sigma_3}$. By Corollary 17, a general pencil X/B will meet the locus of curves with hyperflexes in only finitely many points, so the locus of hyperflexes, and hence the degeneracy locus of the sections $\xi_{\sigma_1}, \xi_{\sigma_2}, \xi_{\sigma_3}$, is a 0-dimensional subscheme of the total space of the pencil. Then, by Theorem 47, the Chow class of this degeneracy locus is given by the Chern class

$$c_2(\mathcal{P}^4_{X/B}(\mathcal{O}_X(1))^{\vee\vee}).$$

As with Proposition 55 for the case of flexes on a single curve, it will later prove fruitful to work not merely with the 4^{th}-order sheaves of invincible parts, but with the m^{th}-order ones, and over an arbitrary line bundle $\mathcal{L}$ on X; we will find use for this more general result in § 4.2.3. By Proposition 72, we have that

$$c_2(\mathcal{P}^m_{X/B}(\mathcal{L})^{\vee\vee}) = \binom{m}{2} \cdot c_1(\mathcal{L})^2 + \left(3\binom{m+1}{4} - \binom{m}{3}\right) \cdot c_1(\omega_{X/B})^2 + \left(3\binom{m+1}{3} - 2\binom{m}{2}\right) \cdot c_1(\omega_{X/B}) \cdot c_1(\mathcal{L}). \tag{3.6}$$

Unlike in the case of counting flexes on a smooth plane curve, we cannot immediately apply the above result to counting hyperflexes. Indeed, the astute reader would notice that the formula in (3.6) differs from Ran's expression in Theorem 66 by the term $\binom{m+1}{4} \cdot \delta$, so there must be more to the story than merely computing the Chern classes of the sheaves of invincible parts. Furthermore, since the missing term involves δ, the number of nodal fibers, it appears that the Chern class in (3.6) has nonzero support at the singular points of the singular fibers of the family. Because we stipulated that inflection points must be smooth (see Definition 6), we need to excise the "automatic contributions" that these singular points make to the Chern class in (3.6). In the remainder of this section, we make this notion of "automatic contribution" precise.

Defining Automatic Degeneracy

Let X/B be an admissible family, let $\mathcal{L}$ be a line bundle on X, and let $p \in X$ be a singular point of a fiber. Let $\sigma_1, \dots, \sigma_{m-1} \in \Gamma(\mathcal{L})$ be global sections, and suppose that the degeneracy locus V of the corresponding global sections $\xi_{\sigma_1}, \dots, \xi_{\sigma_{m-1}} \in \Gamma(\mathcal{P}^m_{X/B}(\mathcal{L})^{\vee\vee})$ is 0-dimensional. Then the "automatic contribution" of V at p is given by

$$\dim_k \mathcal{O}_{V,p} = \dim_k \widehat{\mathcal{O}}_{V,p} = \dim_k \widehat{\mathcal{O}}_{X,p}/I_V,$$

where I_V is the ideal in $\widehat{\mathcal{O}}_{X,p}$ cutting out V. To compute this dimension, we need to express the equations cutting out V in analytic-local coordinates. We first express the sheaves of principal parts analytically-locally. In the notation of the proof of Proposition 27, we deduce from (2.7) that we have the following isomorphism of $(\mathcal{O}_{X,p} \otimes_{\mathcal{O}_{B,\pi(p)}} \mathcal{O}_{X,p})$-modules:

$$(\pi_2^*\mathcal{L} \otimes \mathcal{O}_{X\times_B X}/\mathcal{I}^m_\Delta)_p = (\mathcal{O}_{X,p} \otimes_{\mathcal{O}_{B,\pi(p)}} \mathcal{O}_{X,p})/I^m(\mathcal{O}_{X,p} \otimes_{\mathcal{O}_{B,\pi(p)}} \mathcal{O}_{X,p}),$$

where we have made the identification $\mathcal{L}_p \simeq \mathcal{O}_{X,p}$ in the right tensor factor (a valid move because $\mathcal{L}$ is locally free). Pushing forward along the projection map π_1 and taking completions, we deduce that

$$\widehat{\mathcal{P}^m_{X/B}(\mathcal{L})_p} = \big(\widehat{\mathcal{O}}_{X,p} \otimes_{\widehat{\mathcal{O}}_{B,\pi(p)}} \widehat{\mathcal{O}}_{X,p}\big) / I^m \big(\widehat{\mathcal{O}}_{X,p} \otimes_{\widehat{\mathcal{O}}_{B,\pi(p)}} \widehat{\mathcal{O}}_{X,p}\big), \tag{3.7}$$

where we now regard the right-hand side as an $\widehat{\mathcal{O}}_{X,p}$-module via action on the left tensor factor.

Now suppose that the singular fiber containing p is cut out analytically locally by some element $f \in \widehat{\mathcal{O}}_{X,p}$. By making the identification $\widehat{\mathcal{O}}_{X,p} \simeq R := k[[x,y]]$, we may think of f as being a power series in the variables x, y; note that because p is an isolated singularity, we have $\gcd\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = 1$. We can then choose an identification $\widehat{\mathcal{O}}_{B,\pi(p)} \simeq k[[t]]$ with the property that the map on completed local rings induced by the morphism $\pi\colon X \to B$ is given by the obvious map $k[[t]] \to R[[t]]/(f-t) \simeq R$. Using analytic-local coordinates x, y for the left tensor factor of (3.7) and u, v for the right tensor factor, we obtain the following:

$$\begin{aligned} P^m(f) := \widehat{\mathcal{P}^m_{X/B}(\mathcal{L})_p} &\simeq \big(k[[x,y,t]]/(f(x,y)-t) \otimes_{k[[t]]} k[[u,v,t]]/(f(u,v)-t)\big)/ \\ &\qquad (x\otimes 1 - 1\otimes u, y \otimes 1 - 1 \otimes v)^m \\ &\simeq R[[u,v]]/(f(u,v)-f(x,y),(u-x,v-y)^m). \end{aligned}$$

Now that we know what the principal parts sheaf looks like analytically locally, we can write down the sections τ_{σ_n} for each $n \in \{1,\dots,m-1\}$. Indeed, if the analytic-local germ of σ_n is given by

$$\sigma_n = \sum_{i,j\geq 0} a_{ij}^{(n)} \cdot x^i y^j \in R \simeq \widehat{\mathcal{O}}_{X,p},$$

then we have that

$$\tau_{\sigma_n} = \sum_{i,j\geq 0} a_{ij}^{(n)} \cdot u^i v^j \in P^m(f). \tag{3.8}$$

To work out what ξ_{σ_n} is, we need to describe the dual and double dual of the sheaves of principal parts analytically locally. From the proof of Theorem 68, we know that $\mathcal{P}^m_{X/B}(\mathcal{L})^\vee$ is locally free, so

$$P^m(f)^\vee = \widehat{\mathcal{P}^m_{X/B}(\mathcal{L})^\vee_p}$$

is a free R-module of rank m. Choose a basis $(e_0,\dots,e_{m-1})$ of $P^m(f)^\vee$; then the dual elements $(e_0^\vee, \ldots, e_{m-1}^\vee)$ form a basis for the rank-m free R-module

$$P^m(f)^{\vee\vee} = \widehat{\mathcal{P}^m_{X/B}(\mathcal{L})^{\vee\vee}}_p.$$

With this notation, the canonical map $\mathrm{can}_{\mathrm{ev}}$ acts as follows:

$$\mathrm{can}_{\mathrm{ev}} \colon \tau_{\sigma_n} \in P^m(f) \to \sum_{i=0}^{m-1} e_i(\tau_{\sigma_n}) \cdot e_i^\vee \in P^m(f)^{\vee\vee}.$$

It then follows from Definition 21 that $I(V)$ is the ideal in R generated by the $(m-1) \times (m-1)$ minors of the matrix $M(V)$ defined as follows:

$$M(V) := \begin{bmatrix} e_1(\tau_{\sigma_1}) & \cdots & e_1(\tau_{\sigma_{m-1}}) \\ e_2(\tau_{\sigma_1}) & \cdots & e_2(\tau_{\sigma_{m-1}}) \\ \vdots & \ddots & \vdots \\ e_m(\tau_{\sigma_1}) & \cdots & e_m(\tau_{\sigma_{m-1}}) \end{bmatrix} \tag{3.9}$$

In general, it is difficult to explicitly express the global sections $\sigma_1, \ldots, \sigma_{m-1}$ analytically locally. Thus, we cannot easily compute the minors of $M(V)$; to make things worse, it is entirely possible that different tuples of global sections $\sigma_1, \ldots, \sigma_{m-1}$ will have different corresponding "automatic contributions" $\dim_k R/I(V)$. Given this problem, it is natural to ask what the minimum possible "automatic contribution" is over all choices of original sections. In this regard, we make the following definition.

Definition 75. Given an m-tuple of power series $\overline{g} = (g_1, \ldots, g_{m-1}) \in R^{m-1}$, let $M^m_{\overline{g}}$ be the corresponding matrix as in (3.9), and let $I^m_{\overline{g}}$ be the ideal generated by the $(m-1) \times (m-1)$ minors of $M^m_{\overline{g}}$. The m^{th}**-order automatic degeneracy of** f is defined to be

$$\mathrm{AD}^m(f) := \min_{\overline{g} \in R^{m-1}} \dim_k R/I^m_{\overline{g}}.$$

From Definition 75, it is evident that the automatic degeneracy is an invariant associated to the singularity, in the sense that it only depends on the power series f defining the singularity at p. We are naturally led to ask the following question.

Motivating Question 76. How is the automatic degeneracy $\mathrm{AD}^m(f)$ related to other invariants of singularities (e.g., the Milnor number μ_f)?

We provide a partial answer to Motivating Question 76 in § 3.4.3.

Remark 77. The careful reader might observe that automatic degeneracy, as posited in Definition 75, may not actually be relevant to solving global problems like counting hyperflexes in a pencil of plane curves. Indeed, it is unclear whether there exist *global* sections $\sigma_1, \dots, \sigma_{m-1}$ of the line bundle $\mathcal{L}$ with the property that the associated length of degeneracy $\dim_k R/I(V)$ is actually equal to the minimum of $\dim_k R/I_g^m$ over all *analytic-local germs* of sections at the singularity. All that we can be certain of is that the automatic degeneracy is a lower bound on the actual degeneracy of (any choice of) global sections. In subsequent sections, when we apply the notion of automatic degeneracy to work out the number of hyperflexes in a pencil of plane curves and other examples, we shall just assume that the degeneracy of generally chosen global sections is given by the automatic degeneracy.

Fortunately, the above issue is not as disappointing as it first appears to be. Many enumerative formulas proven in the field of intersection theory have similar deficiencies, in the sense that they depend on assumptions (e.g., that the degeneracy scheme is reduced, or that the sections vanish in the right codimension) that are not always straightforward to verify in a particular example. Although we were able to verify these kinds of assumptions in the context of counting flexes, they are harder to check in more complicated examples. For instance, the author is unaware of how to verify that Ran's Chern class computation in Theorem 66 actually gives the correct answer for the number of hyperflexes in a pencil of plane curves. As S. Kleiman put it in [Kle85], intersection theory is "the system of assumptions, accepted principles, and rules of procedure devised to analyze, predict, or otherwise explain the nature or behavior of" intersections of schemes, suggesting that it is a commonly accepted practice to make reasonable assumptions about the validity of enumerative formulas.[4]

3.4 Calculating Automatic Degeneracies

With regards to actually calculating automatic degeneracies for use in enumerative applications, there is good news and bad news. The good news is that the calculations are essentially algorithmic and can potentially be implemented in a computer program. Indeed, given a value for m and a choice of $f \in k[x, y]$, one needs to execute the following procedure to compute the associated automatic degeneracy $AD^m(f)$:

[4] The original quote is actually due to W. Morris and appears in his dictionary [Mor71].

(a) Find a basis for $P^m(f)^{\vee\vee}$;

(b) Compute the minors of the matrix $M^m(f)$ representing the map $\xi^{\oplus(m-1)}$ with respect to the basis found in step (a); and

(c) Compute $AD^m(f) = \dim_k R/I^m(f)$, where $I^m(f)$ is the ideal generated by the minors found in step (b).

The only aspect of the above three-step procedure that is difficult to perform via computer is step (a). Indeed, it appears to be quite challenging to execute the above procedure if we allow m to vary, while keeping f fixed. In other words, given a choice of f, we can use the above procedure compute $AD^m(f)$ for any particular value of m, but it is far more difficult to determine $AD^m(f)$ as a function of m in this way.

3.4.1 The Nodal Case

Nevertheless, something magical happens in the case where $f(x, y) = xy$, so that the associated singularity is nodal. In this section, we show that it is actually possible to determine $AD^m(xy)$ explicitly as a function of m.

Theorem 78. *Let $f(x, y) = xy$. Then we have*

$$AD^m(xy) = \binom{m+1}{4}.$$

Proof. The first step is to find a basis for $P^m(xy)^{\vee\vee}$ that is "nice enough" to render the calculation of $AD^m(xy)$ feasible for all m.

Step (a): Finding a Basis of $P^m(xy)^{\vee\vee}$

As described in § 3.3.2, our approach to finding a basis of $P^m(xy)^{\vee\vee}$ is to first find a basis of $P^m(xy)^{\vee}$ and then take the dual basis. The following lemma tells us that functionals in $P^m(xy)^{\vee}$ satisfy a handy property.

Lemma 79. *Let m be a positive integer, and let $\phi \in P^m(xy)^{\vee}$. For every $i \in \{0, \dots, m\}$, we have $x^i \mid \phi(u^i)$ and $y^i \mid \phi(v^i)$.*

Proof. The lemma is obvious when $i = 0$. For convenience, let the relation $(u-x)^{m-i}(v-y)^i$ be denoted by R_i for each $i \in \{0, \dots, m\}$. Next, observe that every term other than

$(-x)^{m-1} \cdot v$ in relation R_1 contains a factor of y, so

$$y \mid \phi(R_1 - (-x)^{m-1} \cdot v) = \phi(0 - (-x)^{m-1} \cdot v) = (-x)^{m-1} \cdot \phi(v).$$

It follows that $y \mid \phi(v)$, so the lemma holds when $i = 1$. Further observe that every term other than $(-x)^{m-2} \cdot v^2$ in relation R_2 either contains a factor of y^2 or contains a factor of $y \cdot v$, so

$$y^2 \mid \phi(R_2 - (-x)^{m-2} \cdot v^2) = \phi(0 - (-x)^{m-2} \cdot v^2) = (-x)^{m-2} \cdot \phi(v^2).$$

It follows that $y^2 \mid \phi(v^2)$, so the lemma holds when $i = 2$. Continuing in this manner by inductively assuming that, for some $j \in \{0, \dots, m-1\}$, the lemma holds for every $i \in \{0, \dots, j\}$, one can use relation R_{j+1} to deduce that $y^{j+1} \mid \phi(v^{j+1})$. It follows that $y^i \mid \phi(v^i)$ for every $i \in \{0, \dots, m\}$. Since the setup is symmetric under $(x, u) \leftrightarrow (y, v)$, the same argument demonstrates that $x^i \mid \phi(u^i)$ for every $i \in \{0, \dots, m\}$. □

In the next lemma, we use Lemma 79 to construct a basis of $\mathcal{P}^m(xy)^\vee$.

Lemma 80. *For each* $i \in \{0, \dots, m-1\}$, *there exists a unique functional* $e_i \in P^m(xy)^\vee$ *with the following two properties:*

(a) $e_i(u^j) = \delta_{ij} \cdot x^j$ *for each* $j \in \{0, \dots, m-1\}$*; and*

(b) $e_i(v^j)/y^j \in R^*$ *for each* $j \in \{1, \dots, m\}$.

Moreover, the list $(e_0, \dots, e_{m-1})$ *forms a basis of* $P^m(xy)^\vee$ *as an* R*-module.*

Proof. Observe that specifying a map $R[[u, v]]/(uv - xy) \to R$ is equivalent to specifying the images of the powers of u and v. For each $i \in \{0, \dots, m-1\}$, let $\widetilde{e}_i \colon R[[u, v]]/(uv - xy) \to R$ be any map satisfying the condition that $\widetilde{e}_i(u^j) = \delta_{ij} \cdot x^j$ for each $j \in \{0, \dots, m-1\}$. In order for $\widetilde{e}_i$ to descend to a map $e_i \colon P^m(xy) \to R$, the condition $\widetilde{e}_i(R_\ell) = 0$ must be satisfied for each $\ell \in \{0, \dots, m\}$. We claim that the condition $\widetilde{e}_i(R_0) = 0$ merely serves to specify the value of $\widetilde{e}_i(u^m)$. To see why this claim holds, observe that

$$\widetilde{e}_i(R_0) = 0 \iff \sum_{j=0}^{m} \binom{m}{j} \cdot (-1)^{m-j} \cdot x^{m-j} \cdot \widetilde{e}_i(u^j) = 0 \iff$$

$$\widetilde{e}_i(u^m) = \sum_{j=0}^{m-1} \binom{m}{j} \cdot (-1)^{m-j+1} \cdot x^{m-j} \cdot \widetilde{e}_i(u^j) = \binom{m}{i} \cdot (-1)^{m-i+1} \cdot x^m.$$

Thus, $\widetilde{e}_i$ satisfies the condition $\widetilde{e}_i(R_0) = 0$ if and only if $\widetilde{e}_i(u^m)$ is given as above. In much the same manner, the condition $\widetilde{e}_i(R_1) = 0$ determines the value of $b_{i1} = \widetilde{e}_i(v)$; indeed, notice that

$$\begin{aligned}
&\widetilde{e}_i(R_1) = 0 \Longleftrightarrow \\
&(-1)^{m-1} \cdot x^{m-1} \cdot \widetilde{e}_i(v) + \left(\sum_{j=1}^{m-1} \binom{m-1}{j} \cdot (-1)^{m-1-j} \cdot x^{m-j} y \cdot \widetilde{e}_i(u^{j-1}) \right) + \\
&\quad \left(\sum_{j=0}^{m-1} \binom{m-1}{j} \cdot (-1)^{m-j} \cdot x^{m-1-j} y \cdot \widetilde{e}_i(u^j) \right) = 0 \Longleftrightarrow \\
&\widetilde{e}_i(v) = (-1)^i \cdot \binom{m}{i+1} \cdot y.
\end{aligned}$$

We can continue in this manner by using the condition $\widetilde{e}_i(R_{\ell+1})$ and the already-specified values of $\widetilde{e}_i(v^n)$ for $n \in \{1, \ldots, \ell\}$ to solve for $\widetilde{e}_i(v^{\ell+1})$. After much laborious computation, it follows by strong induction that

$$\begin{aligned}
&\widetilde{e}_i(R_\ell) = 0 \text{ for all } \ell \in \{1, \ldots, m\} \Longleftrightarrow \\
&\widetilde{e}_i(v^\ell) = (-1)^i \cdot \frac{\ell(m-i)}{m(\ell+i)} \cdot \binom{m}{i} \cdot \binom{m+\ell-1}{\ell} \cdot y^\ell \text{ for all } \ell \in \{1, \ldots, m\}.
\end{aligned}$$

Notice in particular that $\widetilde{e}_i(v^\ell)/y^\ell \in \mathbb{Z} \setminus \{0\} \subset R^*$ for all choices of i and ℓ. With the values specified as above, the maps $\widetilde{e}_i \colon R[[u,v]]/(uv - xy) \to R$ satisfy the conditions $\widetilde{e}_i(R_\ell) = 0$ and therefore descend to maps $e_i \colon P^m(xy) \to R$. Moreover, since the maps $\widetilde{e}_i$ satisfy points (a) and (b) in the statement of the lemma, so do the maps e_i. Finally, because the elements u^ℓ, v^ℓ for $\ell \in \{0, \ldots, m\}$ generate $P^m(xy)$, and because we have specified the values of $e_i(u^\ell)$ and $e_i(v^\ell)$ for each ℓ, it follows that we have completely determined the maps e_i.

It is evident that the list $(e_0, \ldots, e_{m-1})$ is linearly independent, so it remains to check that this list spans all of $P^m(xy)^\vee$. Let $\phi \in P^m(xy)^\vee$ be any element; observe by Lemma 79 that there exist $a_i \in R$ such that $\phi(u^i) = a_i \cdot x^i$ for each $i \in \{0, \ldots, m-1\}$. Then the functional $\psi = \phi - \sum_{i=0}^{m-1} a_i \cdot e_i \in P^m(xy)^\vee$ has the property that $\psi(u^i) = 0$ for every $i \in \{0, \ldots, m-1\}$. Inductively tracing through the relations $\psi(R_i) = 0$ as we did in the previous paragraph, we deduce that $\psi(u^m) = 0$ and that $\psi(v^\ell) = 0$ for every $\ell \in \{1, \ldots, m\}$, so in fact ψ is the zero functional, and we have $\phi = \sum_{i=0}^{m-1} a_i \cdot e_i$, implying that the list $(e_0, \ldots, e_{m-1})$ spans all of $P^m(xy)^\vee$. □

By taking the dual basis of the basis constructed in Lemma 80, we obtain a basis for the double-dual module $\mathcal{P}^m(xy)^{\vee\vee}$.

Corollary 81. *For each* $i \in \{0, \dots, m-1\}$*, let* $e_i^\vee \in P^m(xy)^{\vee\vee}$ *be the functional defined by* $e_i^\vee(e_j)\delta_{ij}$ *for every* $j \in \{0, \dots, m-1\}$*. Then the list* $(e_0^\vee, \dots, e_{m-1}^\vee)$ *forms a basis of* $P^m(xy)^{\vee\vee}$*.*

Step (b): Computing the Minors

Now, let $\overline{g} = (g_1, \dots, g_{m-1}) \in R^{m-1}$ be a collection of analytic-local germs of $\mathcal{O}_X$ such that the corresponding germs $\xi_{g_1}, \dots, \xi_{g_{m-1}} \in P^m(xy)^{\vee\vee}$ achieve minimal degeneracy in the sense of Definition 75. For each $n \in \{1, \dots, m-1\}$, we write

$$g_n(x, y) = \sum_{i,j\geq 0} a_j^{(n)} \cdot x^i y^j.$$

With this notation, we have from (3.8) that

$$\tau_{g_n} = \sum_{i,j\geq 0} a_{ij}^{(n)} \cdot u^i v^j = \sum_{j,k\geq 0} (xy)^j \cdot \left(a_{(j+k)j}^{(n)} \cdot u^k + a_{j(j+k)}^{(n)} \cdot v^k\right).$$

Applying the functional e_i to τ_{g_n} is somewhat painful, because we need to use the relations R_0 and R_m to respectively express u^k and v^k for $k \geq m$ in terms of smaller powers of u and v in order to apply the result of Lemma 80. It ends up being far too cumbersome to write out $e_i(\tau_{g_n})$ explicitly, but it is not hard to see that there exist units $\alpha_k^{(n)}, \beta_k^{(n)} \in R^*$ that satisfy the following two properties:

(a) The constant terms of $\alpha_k^{(n)}, \beta_k^{(n)}$ are respectively given by $a_{k0}^{(n)}, a_{0k}^{(n)}$;

(b) We have that $e_i(\tau_{g_n})$ is given by

$$e_i(\tau_{g_n}) = \alpha_i^{(n)} \cdot x^i + \sum_{k=1}^{m-1} b_{ik}\beta_k^{(n)} \cdot y^k,$$

Substituting the above result into (3.9), we deduce that the matrix $M_{\overline{g}}^m(xy)$ is given by

$M_g^m(xy) =$

$$\begin{bmatrix} \alpha_0^{(1)} + \sum_{k=1}^{m-1} b_{1k}\beta_k^{(1)} \cdot y^k & \cdots & \alpha_0^{(m-1)} + \sum_{k=1}^{m-1} b_{1k}\beta_k^{(m-1)} \cdot y^k \\ \alpha_1^{(1)} \cdot x + \sum_{k=1}^{m-1} b_{2k}\beta_k^{(1)} \cdot y^k & \cdots & \alpha_1^{(m-1)} \cdot x + \sum_{k=1}^{m-1} b_{2k}\beta_k^{(m-1)} \cdot y^k \\ \vdots & \ddots & \vdots \\ \alpha_{m-1}^{(1)} \cdot x^{m-1} + \sum_{k=1}^{m-1} b_{mk}\beta_k^{(1)} \cdot y^k & \cdots & \alpha_{m-1}^{(m-1)} \cdot x^{m-1} + \sum_{k=1}^{m-1} b_{mk}\beta_k^{(m-1)} \cdot y^k \end{bmatrix}$$

For each $i \in \{1, \dots, m\}$, let Ξ_i denote the $(m-1) \times (m-1)$ minor of $M_g^m(xy)$ obtained by computing the determinant of the matrix that results from deleting the $(m-i+1)^{\text{th}}$ row of $M_g^m(xy)$. The ideal $I_g^m(xy)$ is defined to be the ideal generated by the Ξ_i's, so we need to be able to understand these minors. However, since we were unable to give an explicit description of the coefficients $\alpha_k^{(n)}$ and $\beta_k^{(n)}$ that appear in the entries of $M_g^m(xy)$, we shall consequently be unable to determine the Ξ_i's explicitly. On the bright side, it is possible to provide a description of the Ξ_i's that is adequate for the purpose of computing the automatic degeneracy $AD^m(xy)$. In the following "aside," we introduce a convenient system of representing elements of R that will allow us to obtain such an adequate description of the Ξ_i's.

Aside: "Root Expansions" of Power Series

The space $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ of pairs of nonnegative integers forms a partially ordered set under the relation $(i,j) \leq (i',j')$ if and only if $i \leq i'$ and $j \leq j'$, with equality if and only if $i = i'$ and $j = j'$. We make use of this structure in the next lemma:

Lemma 82. *Let $h = \sum_{i,j\geq 0} a_{ij} \cdot x^i y^j \in R$ be nonzero. There exists a unique finite subset $\Sigma \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, along with units $r_{ij} \in R^*$ for each $(i,j) \in \Sigma$ that are not necessarily unique, such that the following conditions are satisfied:*

(a) $(i,j) \not\leq (i',j')$ *for all* $(i,j), (i',j') \in \Sigma$*; and*

(b) $h = \sum_{(i,j)\in\Sigma} r_{ij} \cdot x^i y^j$.

Proof. Uniqueness, as is often the case, holds trivially. If uniqueness fails, so that we have two distinct such sets Σ and Σ', then for any $(i,j) \in \Sigma \setminus \Sigma'$, it would be possible to express 0 as a sum with the coefficient of the $x^i y^j$ term being nonzero, an absurdity.

As for existence, it suffices to show that we can reduce to the case where h is expressible as a finite sum of distinct monomials in x and y with coefficients in R^*. Indeed, the lemma is obvious given such an expression of h, for one can simply induct on the number of terms in the sum. We now demonstrate that we can reduce to this case. Let $a_{ij} \cdot x^i y^j$ be a (nonzero) term of h having minimal degree, and let $c_{ij} = \sum_{(i',j') \geq (i,j)} a_{i'j'} \cdot x^{i'-i} y^{j'-j}$. Then, for each $n \in \{0, \dots, j-1\}$, let i_n be the smallest among all i' with the property that $a_{i'n} \neq 0$, and let $c'_n = \sum_{i' \geq i_n} a_{i'n} \cdot x^{i'-i_n}$. Similarly, for each $n \in \{j+1, \dots, i+j\}$, let j_n be the smallest among all j' with the property that $a_{nj'} \neq 0$, and let $c''_n = \sum_{j' \geq j_n} a_{nj'} \cdot y^{j'-j_n}$. We then have that

$$h = \left(\sum_{n=0}^{j-1} c'_n \cdot x^{i_n} y^n \right) + (c_{ij} \cdot x^i y^j) + \left(\sum_{n=j+1}^{i+j} c''_n \cdot x^n y^{j_n} \right),$$

where $c_{ij} \in R^*$ and $c'_n, c''_n \in R^*$ for each $n \in \{0, \dots, i+j\} \setminus \{j\}$. We have thus expressed h as a finite sum of distinct monomials in x and y with coefficients in R^*, which is the desired form. $\square$

Definition 83. With notation as in Lemma 82, we say that Σ is the set of **roots** of h and that the expression of h in point (b) is the **root expansion** of h.

The choice of terminology in Definition 83 is motivated by the fact that one can visualize the partially ordered set $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ as a directed graph and the nonzero terms of h as a directed subgraph; then the roots of h are simply those nodes that have no parents on the subgraph corresponding to h.

Back to Step (b): Computing the Minors

We now express the minors Ξ_i in terms of their root expansions.

Proposition 84. *For each $i \in \{1, \dots, m\}$, the root expansion of Ξ_i is given by*

$$\Xi_i = \sum_{j=0}^{m-2} \gamma_{ij} \cdot x^{\kappa_{m-2-j} + \max\{0, i-j-1\}} y^{\kappa_j},$$

*where $\gamma_{ij} \in R$ for each i, j and $\kappa_n = \sum_{i=0}^{n} i$ is the n^{th} **triangular number** for each n.*

Proof. Fix $i \in \{1, \ldots, m\}$. To get at the roots of Ξ_i, we ask the following question: for every nonnegative integer ℓ, what is the smallest j so that Ξ_i has a nonzero term proportional to $x^j y^\ell$? Before we answer this question in full generality, let us work out the argument in the easiest case, namely when $\ell = 0$. For this case, we want to compute the smallest power of x that appears as a term in Ξ_i; this smallest power is evidently the same as that which arises from computing the following determinant, obtained by deleting all nonzero powers of y from the entries of the minor defining Ξ_i:

$$\left|\begin{array}{ccc} \alpha_0^{(1)} & \cdots & \alpha_0^{(m-1)} \\ \alpha_1^{(1)} \cdot x & \cdots & \alpha_1^{(m-1)} \cdot x \\ \vdots & \ddots & \vdots \\ \alpha_{m-1}^{(1)} \cdot x^{m-1} & \cdots & \alpha_{m-1}^{(m-1)} \cdot x^{m-1} \end{array}\right|_{m-i+1} \tag{3.10}$$

where the subscript is meant to indicate that we have deleted the $(m-i+1)^{\text{th}}$ row. It follows by inspection of (3.10) that the smallest j so that Ξ_i has a nonzero term proportional to $x^j y^0$ is $j = \kappa_{m-2} + i - 1$, confirming that $x^{\kappa_{m-2}+i-1}$ is a root of Ξ_i.

Now let us deal with the case when $\ell > 0$. An entry of the minor defining Ξ_i can either contribute a factor of x through the term $\alpha_k^{(n)} \cdot x^k$ or contribute a factor of y through one of the terms $\beta_k^{(n)} \cdot y^k$. Since we are looking for the smallest j so that Ξ_i has a nonzero term proportional to $x^j y^\ell$, we want the y-factors to come from the bottom-most rows of the minor defining Ξ_i, so that the y-factors are essentially replacing the largest x-factors. But notice that in computing Ξ_i, we cannot choose the same power of y from any two of the bottom-most rows. To see why this claim is true, consider the matrix obtained from $M_g^m(xy)$ by deleting all powers of x, and compute any 2×2 minor of it:

$$\left|\begin{array}{cc} \sum_{k=1}^{m-1} b_{ak}\beta_k^{(c)} \cdot y^k & \sum_{k=1}^{m-1} b_{ak}\beta_k^{(c')} \cdot y^k \\ \sum_{k=1}^{m-1} b_{a'k}\beta_k^{(c)} \cdot y^k & \sum_{k=1}^{m-1} b_{a'k}\beta_k^{(c')} \cdot y^k \end{array}\right| =$$

$$\left(\sum_{k=1}^{m-1} b_{ak}\beta_k^{(c)} \cdot y^k\right)\left(\sum_{k=1}^{m-1} b_{a'k}\beta_k^{(c')} \cdot y^k\right) - \left(\sum_{k=1}^{m-1} b_{ak}\beta_k^{(c')} \cdot y^k\right)\left(\sum_{k=1}^{m-1} b_{a'k}\beta_k^{(c)} \cdot y^k\right) =$$

$$\sum_{\substack{1 \le k,k' \le m-1 \\ k \ne k'}} \left(b_{ak}b_{a'k'}\beta_k^{(c)}\beta_{k'}^{(c')} - b_{ak}\beta_k^{(c')}b_{a'k'}\beta_{k'}^{(c)}\right) \cdot y^{k+k'},$$

where in the last step above, we could restrict the sum by stipulating that the indices k and k' be different because the summand evidently vanishes when we set $k = k'$. In other words, choosing the same power of y from any two rows yields a contribution of 0. It follows that the only possible values of ℓ are the triangular numbers κ_j for each $j \in \{0, \dots, m-2\}$, and it further follows that the smallest j so that Ξ_i has a nonzero term proportional to $x^j y^\ell$ is $j = \kappa_{m-2-j} + \max\{0, i-j-1\}$, which confirms that the $x^{\kappa_{m-2-j}+\max\{0,i-j-1\}} y^{\kappa_j}$ is a root of Ξ_i. Thus, we have the proposition. □

Remark 85. It is not easy to determine the actual expressions for γ_{ij} from the proof of Proposition 84, but doing so is unnecessary for our purposes. All we need is the following fact: the constant term of each γ_{ij} is a polynomial in the constant terms of the coefficients $\alpha_k^{(n)}, \beta_k^{(n)}$ for $k \in \{0, \dots, m-1\}$. Recall that we established in Step (b) that the constant term of $\alpha_k^{(n)}$ is $a_{k0}^{(n)}$ and the constant term of $\beta_k^{(n)}$ is $a_{0k}^{(n)}$. Thus, each γ_{ij} is a polynomial in the coefficients $a_{k0}^{(n)}$ and $a_{0k}^{(n)}$ for $k \in \{0, \dots, m-1\}$ of the original analytic-local germs $g_1, \dots, g_{m-1}$. In particular, the constant terms of all of the γ_{ij}'s depend on only finitely many of the coefficients of of the germs $g_1, \dots, g_{m-1}$.

Finishing Up

We are now ready to combine the results from previous steps to compute the automatic degeneracy $\mathrm{AD}^m(xy)$.

Lemma 86. *We have that*

$$\dim_k R/I_g^m(xy) \geq \dim_k R/(\{x^{\kappa_{m-2-j}} y^{\kappa_j} : j \in \{0, \dots, m-2\}\}),$$

with equality achieved for a general choice of $a_{k0}^{(n)}$ *and* $a_{0k}^{(n)}$ *for* $k \in \{0, \dots, m-1\}$.

Proof. We know that $I_g^m(xy)$ is generated by the minors $\Xi_1, \dots, \Xi_m$ whose root expansions we determined in Proposition 84. Observe that for each $i \in \{1, \dots, m-1\}$, the relation $\Xi = 0$ can be expressed as a relation on the monomials $x^{\kappa_{m-2-j}} y^{\kappa_j}$ for each $j \in \{0, \dots, m-2\}$ with coefficient given by $\gamma_{ij} \cdot x^{\max\{0,i-j-1\}}$. Thus, we may view the relations $\Xi_i = 0$ for $i \in \{1, \dots, m-1\}$ as a system of equations in the variables $x^{\kappa_{m-2-j}} y^{\kappa_j}$; putting this system into matrix form yields

$$\begin{bmatrix} \gamma_{10} & \gamma_{11} & \cdots & \gamma_{1(m-3)} & \gamma_{1(m-2)} \\ x \cdot \gamma_{20} & \gamma_{21} & \cdots & \gamma_{2(m-3)} & \gamma_{2(m-2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{m-3} \cdot \gamma_{(m-2)0} & x^{m-4} \cdot \gamma_{(m-2)1} & \cdots & \gamma_{(m-2)(m-3)} & \gamma_{(m-2)(m-2)} \\ x^{m-2} \cdot \gamma_{(m-1)0} & x^{m-3} \cdot \gamma_{(m-1)1} & \cdots & x \cdot \gamma_{(m-1)(m-3)} & \gamma_{(m-1)(m-2)} \end{bmatrix} \cdot \begin{bmatrix} x^{\kappa_{m-2}} \\ x^{\kappa_{m-3}} y^{\kappa_1} \\ \vdots \\ x^{\kappa_1} y^{\kappa_{m-3}} \\ y^{\kappa_{m-2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

To solve the above system of equations, all we need to do is put the associated augmented matrix into row echelon form. After doing this, the first $m-2$ entries of the last row are 0, so as long as the constant term of $\gamma_{(m-1)(m-2)}$ is nonzero, so that $\gamma_{(m-1)(m-2)}$ is a unit, we deduce that $y^{\kappa_{m-2}} = 0$. Going up one row, the first $m-3$ entries of the second-to-last row are 0, so as long as $\gamma_{(m-2)(m-3)}$ is a unit, we deduce that $x^{\kappa_1}y^{\kappa_{m-3}} = 0$. Continuing inductively in this manner, we deduce that $x^{\kappa_{m-2-j}}y^{\kappa_j} = 0$ as long as $\gamma_{(i+1)i}$ is a unit for each $i \in \{0, \dots, m-2\}$. By Remark 85, this condition on the $\gamma_{(i+1)i}$'s will be satisfied for a general choice of the coefficients $a_{k0}^{(n)}$ and $a_{0k}^{(n)}$ for $k \in \{0, \dots, m-1\}$. Thus, we have shown that

$$R/I_g^m(xy) \simeq R/(\{x^{\kappa_{m-2-j}}y^{\kappa_j} : j \in \{0, \dots, m-2\})$$

as long as the generality condition on the coefficients $a_{k0}^{(n)}$ and $a_{0k}^{(n)}$ is satisfied, and this implies the equality of dimensions in the statement of the lemma. If the generality condition is not satisfied, then not all of the monomials $x^{\kappa_{m-2-j}}y^{\kappa_j}$ may be 0, so the dimension of $R/I_g^m(xy)$ may be larger, giving the desired inequality. □

Given the result of Lemma 86, we are finally ready to compute the automatic degeneracy in the nodal case.

Lemma 87. *We have that*

$$\dim_k R/(\{x^{\kappa_{m-2-j}}y^{\kappa_j} : j \in \{0, \dots, m-2\}) = \binom{m+1}{4}.$$

Proof. Clearly, there is a unique basis of

$$R/(\{x^{\kappa_{m-2-j}}y^{\kappa_j} : j \in \{0, \dots, m-2\})$$

with the property that each basis vector is a monomial in x and y. This basis may be equivalently described as follows: consider the directed graph $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, and remove all nodes that either are equal to or are children of the nodes $(\kappa_{m-2-j}, \kappa_j)$ for $j \in \{0, \dots, m-2\}$. Then the number of nodes that remain in the graph is the desired dimension.

To compute the number of remaining nodes, we sum the number that remain in the "ray" of nodes of the form $(-, \ell)$ over $\ell \in \{0, \dots, \kappa_{m-2} - 1\}$. This sum is most easily computed by splitting it into the chunks $\kappa_j \leq \ell \leq \kappa_{j+1} - 1$ for $j \in \{0, \dots, m-3\}$: indeed, for each ℓ in this interval, the number of nodes that remain in the corresponding "ray" is simply κ_{m-2-j}. Thus, the total number of nodes that remain is simply

$$\begin{aligned}
\sum_{\ell=0}^{\kappa_{m-2}-1} \#(\text{nodes of type } (-,\ell) \text{ that remain}) &= \sum_{j=0}^{m-3} \sum_{\ell=\kappa_j}^{\kappa_{j+1}-1} \kappa_{m-2-j} \\
&= \sum_{j=0}^{m-3} \kappa_{m-2-j} \cdot (\kappa_{j+1} - \kappa_j) \\
&= \sum_{j=0}^{m-3} \frac{(m-2-j)(m-2-j+1)}{2} \cdot (j+1) \\
&= \binom{m+1}{4},
\end{aligned}$$

where in the last step above, we have appealed to the standard identities for summing consecutive squares and cubes to obtain the desired formula. □

Lemmas 86 and 87 together imply that $\mathrm{AD}^m(xy) = \binom{m+1}{4}$. This concludes the proof of Theorem 78. □

3.4.2 Counting Hyperflexes, At Last

We are at last ready to reap the benefits of our efforts to compute the automatic degeneracy in the nodal case by determining the number of hyperflexes in a general pencil of plane curves. To do this, we begin by recalling some notation from the second part of § 3.2.2, where we discussed the results of Ran. Let X/B be an admissible family, and let $\Gamma \subset X$ be the the locus of singular points of fibers of the family. Note that Γ is a finite collection, and suppose for each $p \in \Gamma$ that the analytic-local function cutting out the singularity at p is given by $f_p \in R$. We want to compute the Chern class $c_2(\mathcal{P}^m_{X/B}(\mathcal{L}))$ minus the contributions arising from the singular points; as long as we make the generality assumption stated in Remark 77, we know that the contribution from the singular points is given by

$$\sum_{p \in \Gamma} \mathrm{AD}^m(f_p).$$

If every one of the singular points is nodal and if $\delta = \#(\Gamma)$, then by Theorem 78, we have that the contribution from the singular points is $\binom{m+1}{4} \cdot \delta$, so subtracting this from aforementioned Chern class (which we computed in Proposition 72) yields that

$$\begin{aligned}
&\deg c_2(\mathcal{P}^m_{X/B}(\mathcal{L})) - \sum_{p\in\Gamma} \mathrm{AD}^m(f_p) = \\
&\binom{m}{2}\cdot c_1(\mathcal{L})^2 + \left(3\binom{m+1}{4} - \binom{m}{3}\right)\cdot c_1(\omega_{X/B})^2 + \\
&\quad \left(3\binom{m+1}{3} - 2\binom{m}{2}\right)\cdot c_1(\omega_{X/B})\cdot c_1(\mathcal{L}) - \binom{m+1}{4}\cdot\delta,
\end{aligned} \tag{3.11}$$

which is precisely the formula obtained by Ran in Theorem 66! The fact that we have recovered Ran's formula here suggests that the generality assumption we are relying on is probably valid, although it is not easy to directly prove its validity.

All that remains is to apply the formula to the case where X/B is a pencil of plane curves of degree d and $\mathcal{L} = \mathcal{O}_X(1)$. We first need to provide an explicit construction of the pencil. Clearly the base is $B = \mathbb{P}^1_k$. Now, suppose the pencil is generated by two homogeneous degree-d polynomials $F, G \in k[X_0, X_1, X_2]$. Consider the rational map

$$\pi' \colon \mathbb{P}^2_k \dashrightarrow \mathbb{P}^1_k, \quad [X_0 : X_1 : X_2] \mapsto [F(X_0, X_1, X_2) : G(X_0 : X_1 : X_2)].$$

The map π' is not defined at the common vanishing locus D of F, G; if the pencil is chosen to be sufficiently general, then D is the union of d^2 reduced points. It follows that if we take $X = \mathrm{Bl}_D\,\mathbb{P}^2_k$ to be the blowup of $\mathbb{P}^2_k$ along the locus D, then the rational map π' defined above extends to a morphism $\pi \colon X \to B$. Furthermore, it is clear that the fiber of π above a point $[s : t] \in B = \mathbb{P}^1_k$ is just the vanishing locus in $\mathbb{P}^2_k$ of the homogeneous degree-d polynomial $t\cdot F - s\cdot G$, so the fibers of the family X/B are precisely the curves that constitute the pencil generated by F, G.

Now that we have explicitly constructed our pencil, we need to work out the degrees of the Chern classes $c_1(\mathcal{L})^2, c_1(\mathcal{L})c_1(\omega_{X/B}), c_1(\omega_{X/B})^2 \in A^0(X)$. Let $\phi \colon X \to \mathbb{P}^2_k$ be the map embedding the fibers of the family in the plane (which we now know to be given by the blowdown map $\mathrm{Bl}_D\,\mathbb{P}^2_k \to \mathbb{P}^2_k$). By the compatibility of Chern classes with pullbacks (see part (d) of Theorem 47), we know that

$$c_1(\mathcal{O}_X(1)) = \phi^* c_1(\mathcal{O}_{\mathbb{P}^2_k}(1)) = \phi^*\zeta.$$

We then have that

$$\deg c_1(\mathcal{O}_X(1))^2 = \deg(\phi^*\zeta)^2.$$

By [EH16, part (b) of Proposition 2.19], we have that $(\phi^*\zeta)^2 = \phi^*(\zeta^2)$, so since ζ^2 is the class of a point, it follows that $\deg \phi^*(\zeta)^2 = 1$. Thus, we deduce that

$$\deg c_1(\mathcal{O}_X(1))^2 = \zeta^2 = 1.$$

Let us now work out $c_1(\omega_{X/B})$. We know from [HM98, p. 84] that $\omega_{X/B} = \Omega_X^1 \otimes \pi^*(\Omega_B^1)^\vee$, so part (d) of Theorem 47, Lemma 51, and the result in Remark 52 together tell us that

$$c_1(\omega_{X/B}) = K_X - \pi^* K_B.$$

Note that $K_{\mathbb{P}_k^2} = -3\zeta$ by [EH16, § 1.4.3], so $K_X = K_{\mathbb{P}_k^2} + E = -3 \cdot (\phi^*\zeta) + E$, where E is the class of the exceptional locus. For the same reason, we know that K_B is -2 times the class of a point in the base, so its pullback $\pi^* K_B$ is -2 times the class of a curve in the family, which is $d \cdot (\phi^*\zeta) - E$, so $\pi^* K_B = -2d \cdot (\phi^*\zeta) + 2E$. It follows that

$$c_1(\omega_{X/B}) = (-3 \cdot (\phi^*\zeta) + E) - (-2d \cdot (\phi^*\zeta) + 2E) = (2d-3) \cdot (\phi^*\zeta) - E.$$

Since a general line in $\mathbb{P}_k^2$ fails to meet the locus D that we have blown up, it follows that $(\phi^*\zeta) \cdot E = 0$. Also, by [EH16, part (d) of Proposition 2.19], we know that $\deg E^2 = -\#(E) = -d^2$. Combining the above results, we deduce that

$$\deg c_1(\mathcal{L}) c_1(\omega_{X/B}) = 2d - 3 \quad \text{and} \quad \deg c_1(\omega_{X/B})^2 = (2d-3)^2 - d^2 = 3d^2 - 12d + 9.$$

Substituting these expressions in to our formula (3.11) and using the fact that $\delta = 3(d-1)^2$ from Corollary 62, we find that

$$\begin{aligned} &\deg c_2(\mathcal{P}^m_{X/B}(\mathcal{L})) - \sum_{p \in \Gamma} \mathrm{AD}^m(f_p) = \\ &\frac{1}{4} m(m-1)(12 + 2d(d-5) - 16m + md(17-3d) + m^2(d-1)(d-4)). \end{aligned} \tag{3.12}$$

Substituting in $m = 4$ to obtain the count for hyperflexes gives $6(d-3)(3d-2)$, which agrees with the result from Theorem 64!

3.4.3 Higher-Order Singularities

Repeating the analysis performed in § 3.4.1 for a nodal singularity is next to impossible for higher-order singularities. Indeed, the proof of Theorem 78 involved many laborious computations that were specific to the local equation $f = xy$, and it is unclear as to how to reproduce them for any other choice of f. In this section, we pursue two different approaches to the problem of calculating automatic degeneracies associated to higher-order singularities.

A Direct Approach

Although it is challenging to compute $AD^m(f)$ for arbitrary m and f, the problem becomes somewhat easier if we fix the value of m and the singularity type f. In what follows, we shall provide an "algorithm" for finding a basis of $P^m(f)^{\vee\vee}$ given a specific choice of m and f. It should then be possible — at least theoretically speaking — for a computer to calculate the value of $AD^m(f)$.

Fix a positive integer m and $f \in R$. Recall that

$$P^m(f) \simeq R[[u,v]]/(f(u,v) - f(x,y), (u-x, v-y)^m).$$

To simplify notation a bit, set $a = u - x$ and $b = v - y$. Then the relations $(u-x)^i(v-y)^{m-i} = 0$ can be more compactly written as $a^i b^j = 0$; moreover, we can express the relation $f(u,v) - f(x,y) = 0$ in terms of a, b by Taylor expanding $f(u,v)$ at the point $(u,v) = (x,y)$, which we do as follows:

$$f(u,v) - f(x,y) = \sum_{\substack{i,j \geq 0 \\ i+j>0}} \frac{1}{i!j!} \cdot \frac{\partial^i f}{\partial x^i} \frac{\partial^j f}{\partial y^j} \cdot a^i b^j = 0. \tag{3.13}$$

Now, notice that if we multiply equation (3.13) by $a^i b^{m-2-i}$, we obtain the relations

$$\frac{\partial f}{\partial x} \cdot a^{i+1} b^{m-2-i} + \frac{\partial f}{\partial y} \cdot a^i b^{m-1-i} = 0. \tag{3.14}$$

for any $i \in \{0, \dots, m-2\}$.

Suppose $\phi \in P^m(f)^\vee$ is any functional. One readily checks that the relations in (3.14) imply that

$$\phi(a^i b^{m-1-i}) = c \cdot (-1)^i \cdot \left(\frac{\partial f}{\partial x}\right)^{m-1-i} \left(\frac{\partial f}{\partial y}\right)^i$$

for each $i \in \{0, \dots, m-2\}$, where $c \in R$ is some power series. We now assume that there exists a functional $\phi_m \in P^m(f)^\vee$ satisfying $c = 1$. To prove that this assumption holds for our given choice of m and f, we would have to multiply the relation in (3.13) by monomials in a, b of degree less than or equal to $m-3$ and determine the resulting conditions on the values of $\phi(a^i b^j)$ for $i+j < m-1$. This is precisely what makes the computation of automatic degeneracies for higher order singularities so challenging — it is difficult to solve all of the relations imposed by multiples of (3.14) to determine the existence of ϕ_m. Suppose we have verified that such a map ϕ_i exists for each $i \in \{1, \dots, m\}$. Notice that we can view $\phi_i \in P^i(f)^\vee$ as a functional in $P^m(f)^\vee$ by stipulating that $\phi_i(a^j b^k) = 0$ whenever $j+k \geq i$. It is then easy to check that the list $(\phi_1, \dots, \phi_m)$ forms a basis for $P^m(f)^\vee$; taking the dual basis gives a basis for $P^m(f)^{\vee\vee}$. We can then write down the matrix (which we denoted M_g^m in § 3.3.2) whose degeneracy locus we want to determine, and we can use a computer algebra system like `Macaulay` to compute the length of the vanishing locus of the $(m-1) \times (m-1)$ minors of M_g^m.

In the following example, we illustrate how to execute this process for $m = 4$ and $f = y^2 - x^n$ where $n \in \{3, 4\}$.

Example 88. Suppose we want to determine the number of hyperflexes in an admissible family of curves, where the singular fibers are now allowed to be cuspidal. Recall that a cusp singularity is cut out analytically locally by $f = y^2 - x^3$. We are thus interested in understanding the R-module $P^4(y^2 - x^3)$. By following the "algorithm" described above, it is possible to show that the maps ϕ_i for $i \in \{1, 2, 3, 4\}$ defined as follows form a basis for $P^4(y^2 - x^3)$: letting $g = \sum_{i,j=0}^{3} c_{ij} \cdot a^i b^j \in P^4(y^2 - x^3)$, we have

$$\begin{aligned}
\phi_1(g) &= a_{00}, \\
\phi_2(g) &= a_{10} \cdot 2y + a_{01} \cdot 3x^2, \\
\phi_3(g) &= a_{10} \cdot (2y + 3x^2) + a_{01} \cdot (3x^2 + 6xy) + a_{20} \cdot 4y^2 + a_{11} \cdot 6x^2 y + a_{02} \cdot 9x^4, \\
\phi_4(g) &= a_{01} \cdot 4y^2 + a_{20} \cdot (-12x^2 y) + a_{11} \cdot (9x^4 + 12xy^2) + a_{02} \cdot 36x^3 y + a_{30} \cdot 8y^3 + \\
&\qquad a_{21} \cdot 12x^2 y^2 + a_{12} \cdot 18x^4 y + a_{03} \cdot 27x^6.
\end{aligned}$$

Taking the duals of the above functionals as a basis for $P^4(y^2 - x^3)^{\vee\vee}$ and using `Macaulay` to compute the automatic degeneracy, we find that

$$\mathrm{AD}^4(y^2 - x^3) = 10.$$

We can do the same sort of analysis for the case where the singularity is a tacnode, so that it is cut out by $f = y^2 - x^4$. In that case, one checks that the maps ϕ_i for

$i \in \{1, 2, 3, 4\}$ defined as follows form a basis for $P^4(y^2 - x^4)$:

$$\begin{aligned}
\phi_1(g) &= a_{00}, \\
\phi_2(g) &= a_{10} \cdot 2y + a_{01} \cdot 4x^3, \\
\phi_3(g) &= a_{10} \cdot (4x^3) + a_{01} \cdot (12x^2y) + a_{20} \cdot 4y^2 + a_{11} \cdot 8x^3y + a_{02} \cdot 16x^6, \\
\phi_4(g) &= a_{01} \cdot 16xy^2 a01 + a_{20} \cdot (-16x^3y) + a_{11} \cdot (16x^6 + 24x^2y^2) + a_{02} \cdot 96x^5y + a_{30} \cdot 8y^3 + \\
&\quad a_{21} \cdot 16x^3y^2 + a_{12} \cdot 32x^6y + a_{03} \cdot 64x^9.
\end{aligned}$$

Once again, aking the duals of the above functionals as a basis for $P^4(y^2 - x^4)^{\vee\vee}$ and using `Macaulay` to compute the automatic degeneracy, we find that

$$\mathrm{AD}^4(y^2 - x^4) = 17.$$

Singularities as Limits of Nodes

In Example 88, we showed that $\mathrm{AD}^4(y^2 - x^3) = 10$, but notice that $\mathrm{AD}^4(xy) = \binom{4+1}{4} = 5$. Since $10 = 2 \cdot 5$, it is natural to wonder whether there is any reason to expect the value of $\mathrm{AD}^m(y^2 - x^3)$ to be twice (or if not twice, then some other integer multiple) the value of $\mathrm{AD}^m(xy)$ for each m. This sort of reasoning leads to the following idea: although we cannot directly compute the automatic degeneracy associated to an arbitrary plane curve singularity, maybe we can relate it to the automatic degeneracy of a nodal singularity, which we understand very well. To accomplish this, we introduce the following definition.

Definition 89. Given the analytic-local germ $f \in R$ of an isolated plane curve singularity, we define $\mu_f := R/(f_x, f_y)$ to be the **Milnor number** of f.

It turns out that the Milnor number μ_f measures the "nodality" of a plane curve singularity with analytic-local equation given by f, in the sense that μ_f is the the number of nodes converging to the singularity in the associated versal deformation space. The following corollary of Theorem 78 tells us how the automatic degeneracy associated to a singularity grows with its nodality.

Corollary 90. *Retain the setting of Definition 89. We have the inequality*

$$\mathrm{AD}^m(f) \geq \mu_f \cdot \binom{m+1}{4}.$$

Proof. Let X/S be a family of curves, where $\dim X = 3$ and $\dim S = 2$, and view S as a family over a base B with $\dim B = 1$. Suppose that the fiber of X over the general point of B is an admissible family of curves such that its singular fibers are only nodal, but that the fiber of a point $b_0 \in B$ is an admissible family of curves with some member having a singularity cut out analytic-locally by f. The length of the automatic degeneracy scheme, which is supported at the singular fibers of the composite map $X \to B$, varies upper-semicontinuously along B. In particular, since the number of nodes converging into the singularity cut out by f is the Milnor number μ_f, the automatic degeneracy at the singularity cut out by f is at least as large as the total automatic degeneracy at the nodal singularities limiting toward it, which is the desired result. □

It follows from Corollary 90 that the quantity

$$\mathrm{AD}^m(f) - \mu_f \cdot \binom{m+1}{4} \tag{3.15}$$

is a nonnegative integer function of m that is canonically associated to the isomorphism class of the singularity. We have therefore come across what might just be a new invariant of plane curve singularities, thus providing an answer to Motivating Question 76.

Geometrically, we can interpret the invariant quantity in (3.15) as the number of m^{th}-order inflection points converging at the singularity. In the case of hyperflexes and the tacnode $f = y^2 - x^4$, the Milnor number is $\mu_f = 3$, so the automatic degeneracy needs to be at least $3 \cdot 5 = 15$ by Corollary 90, and accordingly, Example 88 tells us that the automatic degeneracy is 17. The invariant quantity in (3.15) is equal to $17 - 15 = 2$ in this case, so we conclude that a tacnode "counts as two hyperflexes."

To conclude our discussion of automatic degeneracies, we make one final point. In the case of a nodal singularity, the automatic degeneracy was a polynomial in m of degree 4. It is natural to wonder whether a similar such result holds for an arbitrary plane curve singularity, so we pose the following question for future research.

Question 91. *Is the invariant quantity in* (3.15) *a polynomial, or at least eventually a polynomial, in m? If so, what is the degree of this polynomial, and is it always equal to* 4*?*

Chapter 4

Linear Systems and Weierstrass Points

"The purpose of computation is insight, not numbers."

Richard Hamming, 1915–1998

In § 3, we answered some enumerative questions about flexes and hyperflexes of plane curves by computing the Chern classes of the sheaves of principal and invincible parts. However, we ended up computing these Chern classes in much greater generality than our applications required. For example, in the case of counting flexes on a plane curve C, we only ever needed to compute $c_1(\mathcal{P}^3_C(\mathcal{O}_C(1)))$, but we actually worked out $c_1(\mathcal{P}^m_C(\mathcal{L}))$ for all positive integers m and line bundles $\mathcal{L}$ on C. It is therefore natural to ask the following question:

Motivating Question 92. Given the generality in which our Chern class computations apply, can we interpret these Chern classes geometrically, just like we were able to relate $c_1(\mathcal{P}^3_C(\mathcal{O}_C(1)))$ to the number of flexes on C?

The objective of this brief chapter is to demonstrate that the answer to Motivating Question 92 is a resounding yes. To answer it, we introduce a construction called a **linear system**, and we generalize the notion of inflection point as given in Definition 6 to make sense for linear systems, thus providing an answer to Motivating Question 19. Armed with this more general concept of inflection point, we show that the aforementioned Chern classes of the sheaves of principal and invincible parts can be used to shed light on enumerative questions about inflection points of linear systems. We conclude the chapter with a discussion of **Weierstrass points**, which are a particular type of inflection point, and we demonstrate how to use the main results of this thesis to compute certain Chow classes relating to Weierstrass points on the (partially compactified) moduli space of curves.

4.1 Linear Systems

We begin by defining the key object of interest in this chapter, namely the linear system.

Definition 93. Let X be a scheme, let $\mathcal{L}$ be a line bundle on X, and let $W \subset \Gamma(\mathcal{L})$ be a vector space whose elements are global sections of $\mathcal{L}$. The pair $(\mathcal{L}, W)$ is said to be a **linear system** on X.

Let $(\mathcal{L}, W)$ be a linear system on a scheme X. Note that the vanishing locus $V(\sigma)$ of a global section $\sigma \in \Gamma(\mathcal{L})$ is a codimension-1 closed subscheme (i.e., a divisor) of X. Thus, by taking the vanishing loci of sections in W, we can think of the linear system $(\mathcal{L}, W)$ as being a family of divisors of X parameterized by the elements of the projective space $\mathbb{P}_k W$ (since the vanishing locus of a section is invariant with respect to scaling the section). Then the **dimension** of a linear system is defined to be $\dim \mathbb{P}_k W = \dim_k W - 1$. For ease of notation, we let $r = \dim_k W - 1$ in what follows.

Example 94. A 1-dimensional linear system is often called a **pencil**. Recall that back in § 3.2.1, we defined a pencil of degree-d hypersurfaces in $\mathbb{P}_k^n$ to be a line in the projective space parameterizing such hypersurfaces. Since hypersurfaces in $\mathbb{P}_k^n$ are divisors, we can think of a pencil of hypersurfaces as being a family of divisors in $\mathbb{P}_k^n$. Thus, it is natural to ask whether there is a linear system on $\mathbb{P}_k^n$ that gives rise to this family of divisors. The answer is yes: if we take projective coordinates $[X_0 : \cdots : X_n]$ on $\mathbb{P}_k^n$ and let $F, G \in k[X_0, \dots, X_n]$ be homogeneous polynomials of degree d that generate the pencil, consider the 1-dimensional linear system $(\mathcal{L}, W)$ obtained by taking $\mathcal{L} = \mathcal{O}_{\mathbb{P}_k^n}(d)$ and letting $W = \operatorname{span}(F, G) \subset \Gamma(\mathcal{O}_{\mathbb{P}_k^r}(d))$. It is then evident that the vanishing loci of the sections in W form the same family of divisors that our pencil of hypersurfaces does. Thus, the notion of pencil in § 3.2.1 is merely a special case of the notion of pencil introduced in this example.

4.1.1 Inflection Points, Revisited

Now that we have introduced linear systems as a convenient tool for keeping track of families of divisors, we are ready to introduce the generalized notion of inflection point. Let $(\mathcal{L}, W)$ be a linear system on a smooth curve C. Recall from Definition 22 that the order of vanishing of a global section $\sigma \in \Gamma(\mathcal{L})$ along C at a point $p \in C$ is given by $\dim_k \mathcal{O}_{V(\sigma),p}$. We ask the following question: what are all of the possible orders of vanishing of sections in W at the point p? We are thus led to the following definition.

Definition 95. Let C be a curve, let $(\mathcal{L}, W)$ be a linear system on C, and let $p \in C$ be a point. We define the **vanishing sequence** at p associated to $(\mathcal{L}, W)$ to be the (strictly increasing) sequence of nonnegative integers

$$a_0(\mathcal{L}, W, p) < a_1(\mathcal{L}, W, p) < \cdots$$

that occur as orders of vanishing at p of sections in W.

The following lemma demonstrates that vanishing sequences cannot go on forever.

Lemma 96. *Retain the setting of Definition 95. The length of the vanishing sequence is equal to* $\dim_k W = r + 1$.

Proof. Since the order of vanishing of the sum of nonzero sections is the minimum of the orders of vanishing of the sections in the sum, it follows that any collection of sections with the property that each section has a distinct order of vanishing is linearly independent. Thus, the vanishing sequence at p is necessarily finite and bounded in length by $r+1$. On the other hand, we claim that we can choose a basis of $(\sigma_0, \ldots, \sigma_r)$ of W with the property that each σ_i has a distinct order of vanishing, so the length of the vanishing sequence is at least $r+1$ and is therefore equal to $r+1$. One verifies this claim by taking any basis of W and inductively applying the following procedure:

(a) Given two different basis elements τ_1, τ_2 that both vanish to the same order ℓ at p, find a nonzero linear combination $a \cdot \tau_1 + b \cdot \tau_2$, where $a, b \in k^*$, that vanishes to order strictly greater than ℓ at p.[1]

(b) Replace the vectors τ_1, τ_2 with $\tau_1, a \cdot \tau_1 + b \cdot \tau_2$ in the basis. □

[1]Such a linear combination always exists because τ_1, τ_2 are linearly independent and hence not multiples of each other.

Each successive term of a vanishing sequence has to be at least one larger than the previous term, so every such sequence has the property that $a_i(\mathcal{L}, W, p) \geq i$ for each i. It is therefore natural to consider vanishing sequences relative to the "baseline" sequence $(0, 1, 2, \ldots, r)$, and this leads to the next definition.

Definition 97. Retain the setting of Definition 95. We define the **ramification sequence** at p associated to $(\mathcal{L}, W)$ to be the (nondecreasing) sequence of nonnegative integers

$$\alpha_0(\mathcal{L}, W, p) \leq \alpha_1(\mathcal{L}, W, p) \leq \cdots \leq \alpha_r(\mathcal{L}, W, p),$$

where $\alpha_i(\mathcal{L}, W, p) = a_i(\mathcal{L}, W, p) - i$ for each $i \in \{0, \ldots, r\}$.

We are now in position to define inflection points of linear systems.

Definition 98. Retain the setting of Definition 95. The point p is an **inflection point** for $(\mathcal{L}, W)$ if the ramification sequence is not $(0, 0, \ldots, 0)$ (equivalently, if $\alpha_r(\mathcal{L}, W, p) > 0 \iff a_r(\mathcal{L}, W, p) > r$). In particular, p is a **basepoint** for $(\mathcal{L}, W)$ if $\alpha_0(\mathcal{L}, W, p) > 0$. The **weight** of an inflection point p is the sum $\sum_{i=0}^{r} \alpha_i(\mathcal{L}, W, p)$ of the terms in its ramification sequence.

Roughly speaking, an inflection point of a linear system is a point at which the vanishing sequence differs from the "baseline" sequence, so that the ramification sequence fails to consist entirely of zeros; similarly, a basepoint of a linear system is a point at which the vanishing sequence shares no common term with the "baseline" sequence, so that the ramification sequence contains no zeros at all.

If p is not a basepoint of $(\mathcal{L}, W)$, then there exists an open neighborhood $U \subset C$ containing p with the property that no point of U is a basepoint. Let $(\sigma_0, \ldots, \sigma_r)$ be a basis of W as in the proof of Lemma 96. For each point $q \in U$, consider the assignment

$$q \mapsto [\sigma_0(q) : \cdots : \sigma_{\dim_k W - 1}(q)] \in \mathbb{P}_k^r, \tag{4.1}$$

where by $\sigma_i(q)$ we mean the residue (or value) of σ_i at q. We claim that the assignment in (4.1) gives rise to a morphism $\phi \colon U \to \mathbb{P}_k^r$. To see why this claim is true, restrict further to an open subscheme $U' \subset U$ on which $\mathcal{L}$ is trivial. Then the restricted sections $\sigma_i \in \Gamma(U'; \mathcal{L})$ can be viewed as functions on U' via the identification $\Gamma(U'; \mathcal{L}) \simeq \Gamma(U'; \mathcal{O}_C)$. Since not all of the σ_i's vanish at any point of U, it follows (say, by [Vak17, Exercise 6.3.N]) that the assignment in (4.1) defines a morphism $U' \to \mathbb{P}_k^r$.

It is then a simple matter to show that the resulting local morphisms for each $U' \subset U$ on which $\mathcal{L}$ is trivial are independent of the choice of trivialization and glue together into the desired morphism $\phi \colon U \to \mathbb{P}^r_k$.

Consider the line bundle $\mathcal{O}_{\mathbb{P}^r_k}(1)$, and let $\widetilde{\sigma}_i \in \Gamma\big(\mathcal{O}_{\mathbb{P}^r_k}(1)\big)$ be a global section with vanishing locus equal to that of the coordinate X_i for each $i \in \{0, \dots, r\}$. Then it follows from the definition of a pulled-back section that $V(\phi^*\widetilde{\sigma}_i) = V(\sigma_i) \cap U$, so by rescaling the sections $\widetilde{\sigma}_i$ we can arrange that $\phi^*\widetilde{\sigma}_i = \sigma_i$ as elements of $\Gamma(U; \mathcal{L})$ for each $i \in \{0, \dots, r\}$. By Definition 22, the order of vanishing of σ_i along C at p, namely $a_i(\mathcal{L}, W, p)$, is equal to the order of vanishing of $\widetilde{\sigma}_i$ along C at p. In particular, we have the following easy corollary.

Lemma 99. *Retain the setting of Definition 95, and suppose p is not a basepoint. Then $\widetilde{\sigma}_r$ is the unique global section of $\mathcal{O}_{\mathbb{P}^r_k}(1)$ vanishing to highest order along C at p. This highest order is given by $a_r(\mathcal{L}, W, p)$ and is greater than r precisely when p is an inflection point.*

Proof. We have already shown in the preceding discussion that $\widetilde{\sigma}_r$ vanishes to the highest order, namely $a_r(\mathcal{L}, W, p)$, along C at p. The uniqueness statement follows follows from the fact that any global section of $\mathcal{O}_{\mathbb{P}^r_k}(1)$ may be expressed as a linear combination of the sections $\widetilde{\sigma}_i$ and hence has order of vanishing along C at p bounded by the corresponding orders of vanishing of the $\widetilde{\sigma}_i$. The statement about inflection points follows immediately from Definition 98. □

Lemma 99 allows us to provide a geometric interpretation for the statement that p is an inflection point of $(\mathcal{L}, W)$. Indeed, suppose the map $\phi \colon U \to \mathbb{P}^r_k$ restricts to an embedding on some smaller open neighborhood $U' \subset U$ of p. Then, the order of vanishing of $\widetilde{\sigma}_r$ along C at p is (by definition) equal to the intersection multiplicity between the hyperplane $V(\widetilde{\sigma}_r)$ and the local embedding $C \cap U'$ into $\mathbb{P}^r_k$. Thus, inflection points of a linear system on a curve can, under the right conditions, be thought of as points at which a hyperplane in projective space intersects the curve with unusually high intersection multiplicity. This sounds like the notion of inflection point that we started out with in Definition 6; we make this connection explicit in the next section.

4.2 Examples of Inflection Points of Linear Systems

In this section, we illustrate the link between our two notions of inflection point, namely Definitions 6 and 98, through a series of examples.

4.2.1 Flexes on Plane Curves, Once Again

We return once more to the fundamental example of studying flexes on a plane curve. Let C be a smooth plane curve of degree $d \geq 3$, and let $\iota \colon C \hookrightarrow \mathbb{P}^2_k$ denote the embedding of C in the plane. Let $(\sigma_1, \sigma_2, \sigma_3)$ be a basis of $\Gamma(\mathcal{O}_{\mathbb{P}^2_k}(1))$, let $\mathcal{L} = \iota^* \mathcal{O}_{\mathbb{P}^2_k}(1) = \mathcal{O}_C(1)$, and let $W = \operatorname{span}(\iota^*\sigma_1, \iota^*\sigma_2, \iota^*\sigma_3)$.

We claim that the sections $\iota^*\sigma_1, \iota^*\sigma_2, \iota^*\sigma_3$ are linearly independent. Otherwise, if we have $a, b, c \in k$ not all zero such that $a \cdot \iota^*\sigma_1 + b \cdot \iota^*\sigma_2 + c \cdot \iota^*\sigma_3 = 0$, then $\iota^*(a \cdot \sigma_1 + b \cdot \sigma_2 + c \cdot \sigma_3) = 0$; this means that the global section $a \cdot \sigma_1 + b \cdot \sigma_2 + c \cdot \sigma_3$ vanishes on all of C, but this is impossible because the vanishing loci of global sections of $\mathcal{O}_{\mathbb{P}^2_k}(1)$ are lines and we have assumed that C is irreducible of degree at least 3. Thus, we have that W is 3-dimensional, so the vanishing sequence at a point $p \in C$ has three terms. From the discussion immediately following the proof of Lemma 99, we know that p is an inflection point of the linear system $(\mathcal{L}, W)$ if and only if the last term in the vanishing sequence is strictly greater than 2. But this happens if and only if some global section of $\mathcal{O}_{\mathbb{P}^2_k}(1)$ vanishes to order at least 3 along C at p, which is precisely the condition that p is a flex of C. Therefore, we have shown that flexes are inflection points of an appropriately constructed linear system.

Note that the ramification sequence of any point $p \in C$ has the property that its first two terms are 0. Indeed, there is always some line not vanishing at p, which implies that $\alpha_0(\mathcal{L}, W, p) = 0$; moreover, since we took C to be smooth, there is always some line vanishing to order 1 at p, which implies that $\alpha_1(\mathcal{L}, W, p) = 0$ as well. Thus, if p is a weight-n inflection point of $(\mathcal{L}, W)$, then the ramification sequence of p is $(0, 0, w)$. It follows that the flexes of C are the same as the inflection points of weight at least 1.

We can now interpret the Chern class computation in Proposition 55.

Proposition 100. *Let C be as above, and let $(\mathcal{L}, W)$ be a linear system on C of dimension $m - 1$ with the following property: if $(\sigma_1, \ldots, \sigma_m)$ is a basis of W, then the degeneracy locus of $\tau_{\sigma_1}, \ldots, \tau_{\sigma_m}$ is a reduced 0-dimensional subscheme of C. Then the number of inflection points* *of the linear system* $(\mathcal{L}, W)$ *is given by*

$$\deg c_1(\mathcal{P}^m_C(\mathcal{L})) = m \cdot (\deg c_1(\mathcal{L})) + \frac{m(m-1)}{2} \cdot (\deg K_C).$$

As amazing general as Proposition 100 is, the downside is that it may be very difficult to verify the condition that the aforementioned degeneracy locus is reduced. It is natural to hope that this condition holds for a general curve C, but this is not necessarily the case; see [EH16, Exercises 7.40, 7.41] for one counterexample. However, we verified in Lemma 57 that this reducedness condition does hold in the context of counting flexes on C, so at least Proposition 100 can be applied in that case!

To show how Proposition 100 works, we provide two special examples.

Example 101. We can generalize the above discussion of flexes on plane curves to inflection points on curves embedded in projective space of any dimension. Indeed, let $\iota \colon C \hookrightarrow \mathbb{P}^r_k$ be a smooth curve of degree d and genus g. Let $\mathcal{L} = \mathcal{O}_C(1)$, and let W denote the vector subspace of $\Gamma(\mathcal{O}_C(1))$ generated by the pull-backs of global sections via ι of $\mathcal{O}_{\mathbb{P}^r_k}(1)$. Note that Lemma 56 implies the 0-dimensionality assumption required to apply Proposition 100. As long as the reducedness assumption is also satisfied, the number of inflection points of the linear system $(\mathcal{L}, W)$ is given by

$$m \cdot (\deg c_1(\mathcal{O}_C(1))) + \frac{m(m-1)}{2} \cdot (\deg K_C) = md + m(m-1)(g-1),$$

where we have used the values of the degrees of the Chern classes that we found in the aftermath of the proof of Proposition 55.

In the case where $r = 3$ and $m = 4$, the inflection points that arise from this example are called **stall points**; by viewing the curve C as the trajectory of an airplane, a stall point is, quite literally, a point at which the airplane stalls in midair!

Example 102. Let $\iota \colon C \hookrightarrow \mathbb{P}^2_k$ be a smooth plane curve of degree $d \geq 3$. A point $p \in C$ is said to be **sextactic** point if there exists a smooth conic curve $D \subset \mathbb{P}^2_k$ with the property that $\operatorname{mult}_p(C, D) \geq 6$. By using an argument analogous to the one that told us how to write flexes as inflection points of a linear system, it is easy to see that p is sextactic if and only if it is an inflection point of the 5-dimensional linear system $(\mathcal{L}, W)$, where $\mathcal{L} = \mathcal{O}_C(2)$ and W denotes the vector subspace of $\Gamma(\mathcal{O}_C(2))$ generated by the pull-backs of global sections via ι of $\mathcal{O}_{\mathbb{P}^2_k}(2)$. Thus, assuming that the reducedness and 0-dimensionality assumptions are satisfied, the number of sextactic points of C is given by Proposition 100 to be

$$6 \cdot (\deg c_1(\mathcal{O}_C(2))) + \frac{6(6-1)}{2} \cdot (\deg K_C) = 6 \cdot 2d + 15 \cdot 2 \left(\binom{d-1}{2} - 1 \right)$$
$$= 3d(5d-11),$$

where we have used the fact that $\deg c_1(\mathcal{O}_C(2)) = 2d$ (this may be deduced in much the same manner that we deduced $\deg c_1(\mathcal{O}_C(1)) = d$ in the aftermath of the proof of Proposition 55). The number of sextactic points on a general curve C was first computed by A. Cayley in [Cay09, § 341] using purely classical techniques (including the Hessian and other similar constructions), but he obtained $3d(4d-9)$ for his answer, as opposed to our $3d(5d-11)$. What could possibly explain the incongruity between our answer and Cayley's? Well, notice that the difference between the two answers is

$$3d(5d-11) - 3d(4d-9) = 3d(d-2),$$

a number that the astute reader will recognize as the number of flexes on a general curve C. It is then natural to surmise that each of the flexes of C is somehow contributing to our count of sextactic points, and this is in fact true. Indeed, because our method does not distinguish between smooth and singular conics, we end up accounting for the fact that at each flex, the doubled tangent line meets the curve with intersection multiplicity at least 6. Upon eliminating these singular sextactic conics from our count, we arrive at Cayley's formula.

4.2.2 Flexes and Degenerations

Our work with automatic degeneracy also has the power to tell us about the analytic-local geometry of the divisor of weight-1 inflection points of a linear system on a family of curves acquiring a nodal singularity. To see how this works, let X/B be an admissible family of curves in $\mathbb{P}^r_k$ acquiring a nodal singularity at a point $p \in X$ in the fiber $X_{\pi(p)}$. Consider the linear system $(\mathcal{L}, W)$ on X obtained by taking $\mathcal{L} = \mathcal{O}_X(1)$ and W to be the vector subspace of $\Gamma(\mathcal{O}_X(1))$ generated by the pull-backs of global sections of $\mathcal{O}_{\mathbb{P}^r_k}(1)$. The locus of points in X that are weight-1 inflection points of the restrictions of $(\mathcal{L}, W)$ to the fibers of the family is given by the degeneracy locus of $r+1$ general sections of the sheaf of invincible parts $\mathcal{P}^{r+1}_{X/B}(\mathcal{L})^{\vee\vee}$. We want to determine what this degeneracy locus looks like analytically locally near p. But this is easy to do given the calculations made in § 3.4.1. Indeed, it is an easy corollary of Proposition 84 that the power series cutting out the desired degeneracy locus in the ring $R = k[[x, y]]$ has root expansion given by

$$\sum_{i=0}^{r} \gamma_i \cdot x^{\kappa_{r-i}} y^{\kappa_i} \tag{4.2}$$

for some coefficients γ_i that are "general" as long as we make the assumption stated in Remark 77. Interestingly enough, the expression in (4.2) can be factored as follows:

$$\sum_{i=0}^{r} \gamma_i \cdot x^{\kappa_{r-i}} y^{\kappa_i} = \prod_{i=1}^{r} (\alpha_i \cdot x^i - \beta_i \cdot y^{r+1-i})$$

for some coefficients α_i, β_i. We therefore arrive at the following theorem:

Theorem 103. *Retain the above setting. Analytically-locally, the degeneracy locus of* $r + 1$ *general sections of the sheaf of invincible parts* $\mathcal{P}^{r+1}_{X/B}(\mathcal{L})^{\vee\vee}$ *is given by the reduced union of* r ***hypercuspidal*** *branches defined by equations of the form* $y^{r+1-i} = x^i$ *for* $i \in \{1, \dots, r\}$.

Remark 104. The result of Theorem 103 was first proven by S. Cautis in [Cau06, Theorem 3.25], although he used an entirely different approach involving monodromy in families of curves. Our method is more straightforward (in the sense that it is a direct local calculation), if not quite as enlightening!

4.2.3 Hyperflexes in a Pencil, Revisited

Our last example for this section is that of hyperflexes in a pencil of plane curves. Let $X/\mathbb{P}^1_k$ be a pencil of plane curves of degree d, and let $\phi \colon X \to \mathbb{P}^2_k$ denote the morphism whose restriction to each fiber of $\pi \colon X \to \mathbb{P}^1_k$ is the embedding of the fiber in the plane. Let $(\sigma_1, \sigma_2, \sigma_3)$ be a basis of $\Gamma(\mathcal{O}_{\mathbb{P}^2_k}(1))$, let $\mathcal{L} = \phi^*\mathcal{O}_{\mathbb{P}^2_k}(1) = \mathcal{O}_X(1)$, and let $W = \operatorname{span}(\phi^*\sigma_1, \phi^*\sigma_2, \phi^*\sigma_3)$. Then restricting the linear system $(\mathcal{L}, W)$ to any fiber of the pencil gives a linear system on the fiber, the weight-2 inflection points of which are the hyperflexes on that fiber. Indeed, a point $p \in X$ is a weight-2 inflection point if its ramification sequence is $(0, 0, 2)$, but this happens if and only if some global section of $\mathcal{O}_{\mathbb{P}^2_k}(1)$ vanishes to order at least 4 at p along the fiber $X_{\pi(p)}$ containing p, which is precisely the condition that p is a flex of $X_{\pi(p)}$. Therefore, as we did in the case of flexes, we have shown that hyperflexes may be alternatively defined as inflection points, with a certain ramification sequence, of a linear system.

We now use our knowledge of linear systems to apply the formula (3.11) in an interesting example that extends the discussion in Example 102.

Example 105. Let C be a smooth plane curves of degree $d \geq 3$. A point $p \in C$ is said to be a **septactic** point if there exists a smooth conic curve $D \subset \mathbb{P}^2_k$ with the property that $\mathrm{mult}_p(C, D) \geq 7$. It is easy to see that p is septactic if and only if it is an inflection point with ramification sequence $(0, 0, 0, 0, 0, 2)$ of the 5-dimensional linear system $(\mathcal{L}, W)$, where $\mathcal{L} = \mathcal{O}_C(2)$ and W denotes the vector subspace of $\Gamma(\mathcal{O}_C(2))$ generated by the pull-backs of global sections of $\mathcal{O}_{\mathbb{P}^2_k}(2)$.

Now let X/B be a general pencil of plane curves of degree d. By analogy with the fact that a family of curves acquires hyperflexes while individual curves generally only have flexes, we claim that there are finitely many septactic points in the fibers of the family, although individual curves generally only have sextactic points. Assuming that the locus of septactic points is reduced, the number of such points should be given by (3.11), where we take $\mathcal{L} = \mathcal{O}_X(2)$ and $m = 7$. Repeating the calculation of § 3.4.2 with this choice of $\mathcal{L}$ and m yields

$$\deg c_2(\mathcal{P}^7_{X/B}(\mathcal{O}_X(2))) - \binom{m+1}{4} = 315d^2 - 1176d + 693.$$

Similar to what happened in Example 102, the doubled tangent line at a hyperflex meets the curve with intersection multiplicity 8 at the hyperflex, and so each hyperflex contributes to the above formula with multiplicity 2. Thus, we need to subtract off 2 times the number of hyperflexes (see Theorem 64 for this number) to obtain the count of septatic points:

$$693 - 1176d + 315d^2 - 2(6(d-3)(3d-2)) = 9(d-3)(31d-23).$$

4.3 Weierstrass Points

For what remains of this thesis, we turn our attention to a particular type of inflection point known as a Weierstrass point, which we define as follows.

Definition 106. Let n be a positive integer. Let C be a smooth curve, and consider the linear system $(\mathcal{L}, W)$ on C obtained by taking $\mathcal{L} = (\Omega^1_C)^{\otimes n}$ and $W = \Gamma(\mathcal{L})$. An inflection point of $(\mathcal{L}, W)$ is called a **Weierstrass point of order** n.

Remark 107. Weierstrass points play an important role in the study of algebraic curves, and they are particularly useful for understanding automorphisms of such curves because the set of Weierstrass points is carried to itself under any automorphism.

As it happens, it is possible to define Weierstrass points in greater generality; refer to [WL90] for a detailed discussion of how to extend the notion of Weierstrass point in Definition 106 to an arbitrary line bundle $\mathcal{L}$.

4.3.1 Counting Weierstrass Points

Much in the way that a general curve has finitely many flexes and no hyperflexes, we might expect that a general curve has finitely many weight-1 Weierstrass points,[2] and none of higher weight. Moreover, just as a general 1-parameter family of curves has finitely many members with hyperflexes and no 5^{th}-order flexes, we might expect that a general 1-parameter family of curves has finitely many members with weight-2 Weierstrass points, and none of higher weight. The objective of this section is to answer the following two questions concerning Weierstrass points:

(a) How many n^{th}-order Weierstrass points of weight 1 does a curve have; and

(b) How many n^{th}-order Weierstrass points of weight 2 are there in the members of a 1-parameter family of curves?

The Weight-1 Case on an Individual Curve

Let n be a positive integer. Let C be a smooth curve of genus g, let $\mathcal{L} = (\Omega_C^1)^{\otimes n}$, and let $W = \Gamma(\mathcal{L})$. The first thing is to compute the dimension of the linear system $(\mathcal{L}, W)$.

Lemma 108. *With notation as above, we have that*

$$\dim_k W = \begin{cases} g & \text{if } n = 1, \\ (2n-1)(g-1) & \text{if } n > 1 \text{ and } g > 1, \\ 0 & \text{if } n > 1 \text{ and } g = 0, \\ 1 & \text{if } n > 1 \text{ and } g = 1 \end{cases}$$

[2] Recall that flexes have weight 1 and hyperflexes have weight 2.

Proof. When $n = 1$, we know that $\dim W = g$. Now suppose $n > 1$. From Lemma 51, we know that $c_1(\mathcal{L}) = n \cdot c_1(\Omega^1_C) = n \cdot K_C$, so $\deg c_1(\mathcal{L}) = n(2g-2)$. Thus, when $g > 1$, the version of the Riemann-Roch Theorem for line bundles with first Chern class having degree at least $2g-1$ implies that

$$\begin{aligned}\dim_k W &= \deg c_1(\mathcal{L}) + 1 - g \\ &= n(2g-2) + 1 - g = (2n-1)(g-1).\end{aligned}$$

Now, if $g = 0$, then $\deg c_1(\mathcal{L}) = -2n < 0$, which implies that $\mathcal{L}$ cannot have any global sections (because global sections do not have poles by definition), so $\dim_k W = 0$. Finally, if $g = 1$, then $\deg c_1(\mathcal{L}) = 0$; this means that any global section that has no poles must also have no zeros, which further implies that $\mathcal{L}$ is trivial as long as a global section exists. Note that $\dim_k \Gamma(\Omega^1_C) = 1 > 0$, so Ω^1_C — and hence $(\Omega_C)^{\otimes n}$ — must have a global section and must therefore be trivial, yielding that $\dim_k W = 1$. □

The following is an immediate consequence of Lemma 108.

Corollary 109. *Smooth curves of genus* 0 *and* 1 *do not have Weierstrass points of any order.*

Proof. In the genus 0 case, Ω^1_C has no global sections and hence the linear system $(\mathcal{L}, W)$ with respect to which n^{th}-order Weierstrass points are defined does not exist. In the genus 1 case, Ω^1_C is trivial and hence has one global section that does not vanish anywhere on C, implying $(\mathcal{L}, W)$ has no inflection points. □

In light of the result of Corollary 109, we restrict our attention to curves of genus at least 2 throughout what remains of this thesis.

We are now in position to apply Proposition 100 to determine the number of inflection points of the linear system $(\mathcal{L}, W)$, which is of course the number of n^{th}-order Weierstrass points of C. As long as the 0-dimensionality and reducedness assumptions in the statement of the proposition are satisfied, then the desired number is given by

$$(\dim_k W) \cdot (\deg c_1(\mathcal{L})) + \frac{(\dim_k W)((\dim_k W) - 1)}{2} \cdot (\deg K_C) = \\ \begin{cases} g(g-1)(g+1) & \text{if } n = 1, \\ g(g-1)^2(2n-1)^2 & \text{if } n > 1 \end{cases}$$

where we have appealed to the statement and proof Lemma 108 for the values of $\dim_k W$ and $\deg c_1(\mathcal{L})$.

The Weight-2 Case in a Family

Let X/B be an admissible family of curves of genus g. If $C \subset X$ is a smooth fiber of the family, then we have a linear system $(\mathcal{L}, W)$ on C given by taking $\mathcal{L} = (\Omega^1_C)^{\otimes n}$ and $W = \Gamma(\mathcal{L})$. We want to study the weight-2 Weierstrass points of C, but notice that there are two types of weight-2 Weierstrass points: those with ramification sequence $(0, 0, \ldots, 0, 0, 2)$ (which we call **type (a)**) and those with ramification sequence $(0, 0, \ldots, 0, 1, 1)$ (which we call **type (b)**).

Let us first handle the case of type (a) Weierstrass points. Given a point $p \in C$, notice that p is a type (a) Weierstrass point if and only if there exists a global section $\sigma \in W$ vanishing to order $(\dim_k W) + 1$ along C at p. For convenience, let $m = \dim_k W + 1$. If we consider the natural map

$$W \to \Gamma(\Omega^1_C \otimes \mathcal{O}_C/\mathcal{I}^m_p), \tag{4.3}$$

we want the image of σ to be equal to the 0 section. This is now reminiscent of the analysis we performed in § 2.1.1 when were interested in studying flexes on plane curves. The question is: how do we fit the maps in (4.3) together over all fibers of the family? Fortunately, it turns out that the vector bundle $\pi^*(\pi_*\omega_{X/B})$ has the property that its fiber at a point $p \in X$ is given by $\Gamma(\Omega^1_{X_{\pi(p)}})$, and we know that the fiber at p of the principal parts sheaf $\mathcal{P}^m_{X/B}(\omega_{X/B})$ is given by $\Gamma(\Omega^1_C \otimes \mathcal{O}_C/\mathcal{I}^m_p)$, so by fitting together the maps in (4.3) fiber-by-fiber, we obtain the following map of sheaves:

$$\phi \colon \pi^*(\pi_*\omega_{X/B}) \to \mathcal{P}^m_{X/B}(\omega_{X/B}).$$

Of course, $\mathcal{P}^m_{X/B}(\omega_{X/B})$ is not locally free because our family may have singular fibers, but we can apply the strategy developed in § 3.3. To do this, we simply work with the composite map

$$\phi' = \mathrm{can}_{\mathrm{ev}} \circ \phi \colon \pi^*(\pi_*\omega_{X/B}) \to \mathcal{P}^m_{X/B}(\omega_{X/B})^{\vee\vee},$$

which is a map of vector bundles because the sheaves of invincible parts are locally free. The locus of type (a) Weierstrass points is then given by excising the contributions of the singular points of the fibers from the degeneracy locus $D_{m-2}(\phi')$. From Porteous' Formula, we have that the Chow class of $D_{m-2}(\phi')$ is given by

$$\begin{aligned}[D_{m-2}(\phi')] &= c_2([\pi^*(\pi_*\omega_{X/B})]^\vee \otimes \mathcal{P}^m_{X/B}(\omega_{X/B})^{\vee\vee}) \\ &= c_2(\mathcal{P}^m_{X/B}(\omega_{X/B})^{\vee\vee}) + c_2(\pi^*(\pi_*\omega_{X/B})) - c_1(\mathcal{P}^m_{X/B}(\omega_{X/B})^{\vee\vee})c_1(\pi^*(\pi_*\omega_{X/B})).\end{aligned}$$

Now, note that because Chern classes commute with pullbacks and because B is 1-dimensional, we can ignore the term $c_2(\pi^*(\pi_*\omega_{X/B}))$. Combining the above result with our knowledge of the Chern classes of the sheaves of invincible parts from Proposition 72, we find that

$$\begin{aligned}[D_{m-2}(\phi')] &= \left[\binom{m}{2} + \left(3\binom{m+1}{4} - \binom{m}{3}\right) + \left(3\binom{m+1}{3} - 2\binom{m}{2}\right)\right] \cdot c_1(\omega_{X/B})^2 - \\ &\quad \left[m + \binom{m}{2}\right] \cdot c_1(\omega_{X/B}) \cdot c_1(\pi^*(\pi_*\omega_{X/B})) - \sum_{p\in\Gamma} AD^{g+1}(f_p) \\ &= \frac{1}{24}m(m-1)(m+1)(3m+2) \cdot c_1(\omega_{X/B})^2 - \\ &\quad \frac{1}{2}m(m+1) \cdot c_1(\omega_{X/B}) \cdot c_1(\pi^*(\pi_*\omega_{X/B}))\end{aligned}$$

To compute the number of type (a) Weierstrass points in the fibers of the family X/B, all we need to do is take the degree of the class $[D_{m-2}(\phi')]$ computed above and subtract off the automatic degeneracy at the singularities. In this case, the automatic degeneracy is given by $\sum_{p\in\Gamma} AD^m(f_p)$, where the locus of singularities is denoted Γ, and f_p denotes the analytic-local equation of the singularity at $p \in \Gamma$.

We can perform a similar analysis for the case of type (b) Weierstrass points. It is not too hard to show that the number of type (b) Weierstrass points is given by the degree of the Chow class of the degeneracy locus $D_{m-3}(\psi)$, where ψ is the map

$$\psi \colon \pi^*(\pi_*\omega_{X/B}) \to \mathcal{P}^{m-2}_{X/B}(\omega_{X/B})^{\vee\vee}.$$

Applying Porteous' Formula, we deduce that

$$\begin{aligned}[D_{m-3}(\psi)] = c_1([\pi^*(\pi_*\omega_{X/B})]^\vee \otimes \mathcal{P}^{m-2}_{X/B}(\omega_{X/B})^{\vee\vee})^2 - \\ c_2([\pi^*(\pi_*\omega_{X/B})]^\vee \otimes \mathcal{P}^{m-2}_{X/B}(\omega_{X/B})^{\vee\vee}).\end{aligned} \tag{4.4}$$

We have basically already computed the second term of (4.4) while studying type (a)

Weierstrass points. Indeed, one merely needs to make the replacement $m \rightsquigarrow m-2$, which yields that

$$c_2([\pi^*(\pi_*\omega_{X/B})]^\vee \otimes \mathcal{P}^{m-2}_{X/B}(\omega_{X/B})^{\vee\vee}) = \frac{1}{24}(m-2)(m-3)(m-1)(3m-4)\cdot c_1(\omega_{X/B})^2 - \frac{1}{2}(m-2)(m-1)\cdot c_1(\omega_{X/B})\cdot c_1(\pi^*(\pi_*\omega_{X/B})).$$

We now need to compute the first term of (4.4). But this is easy: we have that

$$\begin{aligned}
&c_1([\pi^*(\pi_*\omega_{X/B})]^\vee \otimes \mathcal{P}^{m-2}_{X/B}(\omega_{X/B})^{\vee\vee})^2 = \\
&c_1(\mathcal{P}^{m-2}_{X/B}(\omega_{X/B})^{\vee\vee})^2 - 2\cdot c_1(\mathcal{P}^{m-2}_{X/B}(\omega_{X/B})^{\vee\vee})\cdot c_1(\pi^*(\pi_*\omega_{X/B})) + c_1(\pi^*(\pi_*\omega_{X/B}))^2 = \\
&\left(\frac{1}{2}(m-2)(m-1)\right)^2\cdot c_1(\omega_{X/B})^2 - \\
&\quad (m-2)(m-1)\cdot c_1(\omega_{X/B}\cdot c_1(\pi^*(\pi_*\omega_{X/B})) + c_1(\pi^*(\pi_*\omega_{X/B}))^2.
\end{aligned}$$

Applying Proposition 72 and combining our results, we deduce that

$$[D_{m-3}(\psi)] = \frac{1}{24}m(m-2)(m-1)(3m-5)\cdot c_1(\omega_{X/B})^2 - \frac{1}{2}(m-2)(m-1)\cdot c_1(\omega_{X/B})\cdot c_1(\pi^*(\pi_*\omega_{X/B})) + c_1(\pi^*(\pi_*\omega_{X/B}))^2.$$

Now, it is a simple matter to find the number of type (b) Weierstrass points in the family: all one does is take the degree of the class $[D_{m-3}(\psi)]$ calculated above and subtract off the automatic degeneracy at the singular points of the fibers. In this case, however, the automatic degeneracy at a point $p \in \Gamma$ is not simply given by $\mathrm{AD}^{m-2}(f_p)$. Indeed, the definition and computation of automatic degeneracy performed in § 3.3 and 3.4 is for maps from rank-$(m-1)$ to rank-m vector bundles, but in this case we have a map from a rank-$(m-1)$ to a rank-$(m-2)$ vector bundle. If we denote the new automatic degeneracy by $\widetilde{\mathrm{AD}}^{m-2}(f_p)$, then we need to subtract off $\sum_{p\in\Gamma}\widetilde{\mathrm{AD}}^{m-2}(f_p)$.

Remark 110. By repeating the analysis of the nodal case performed in § 3.4.1, it is possible to show that

$$\widetilde{\mathrm{AD}}^{m-2}(xy) = \binom{m}{4}. \tag{4.5}$$

The argument is a simple modification of that used in the proof § 3.4.1. Indeed, the $(m-2)\times(m-2)$ minors of the matrix corresponding to the map ψ are all of the same form, and assuming a generality condition of the type stated in Remark 77, these minors give $m-1$ relations on the monomials $x^{\kappa_{m-3-j}}y^{\kappa_j}$, so all of these monomials must be in the ideal cutting out the degeneracy scheme. The claimed equality (4.5) then follows from Lemma 87.

4.3.2 Weierstrass Points and The Moduli Space of Curves

In this section, we present an application of the results on classes of Weierstrass points obtained in the previous section to studying the geometry of the moduli space of curves. We provide a very brief summary of the relevant background material before stating the results; for a detailed treatment of the subject, the reader is urged to refer to [HM98].

4.3.3 A Bit of Background

Let $g \geq 2$ be an integer, and let $\overline{\mathcal{M}}_g$ be the **coarse moduli space of stable curves of genus** g (recall that a curve is said to be **stable** if it is complete and connected with only nodes as singularities and has finite automorphism group).[3] It turns out that it is possible to develop a notion of intersection theory on $\overline{\mathcal{M}}_g$, as well as on the space $\overline{\mathcal{C}}_g$, which can be defined in two equivalent ways: (a) as the **universal family** over $\overline{\mathcal{M}}_g$ whose fiber over a point is the corresponding curve, or (b) as the coarse moduli space of stable curves of genus g with a marked point. Indeed, in [Mum83], D. Mumford provides definitions for the Chow rings of $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{C}}_g$, working over $\mathbb{Q}$-coefficients rather than $\mathbb{Z}$-coefficients because the theory becomes simpler. Moreover, he specifies a number of classes in these Chow rings that are useful for studying the geometry of $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{C}}_g$. Of particular interest are certain divisor classes, which Mumford defines as follows: taking $\pi \colon \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ to be the obvious projection map, we have

- $K = c_1(\omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}) \in A^1(\overline{\mathcal{C}}_g) \otimes \mathbb{Q}$;
- $\kappa = \pi_*(K^2) \in A^1(\overline{\mathcal{M}}_g) \otimes \mathbb{Q}$ called the **tautological class**; and
- $\lambda = c_1(\pi_* \omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}) \in A^1(\overline{\mathcal{M}}_g) \otimes \mathbb{Q}$, a class called the **Hodge class**.
- $\delta_i \in A^1(\overline{\mathcal{M}}_g) \otimes \mathbb{Q}$ for $i \in \{0, \dots, \lfloor g/2 \rfloor\}$ corresponding to loci of singular curves: roughly speaking, δ_0 is the class of irreducible curves with a nodal singularity,

[3] The space $\overline{\mathcal{M}}_g$ is called the **Deligne-Mumford** compactification of the moduli space $\mathcal{M}_g$ and was first introduced by the eponymous mathematicians in [DM69].

and δ_i for $i > 0$ is the class of curves with a nodal singularity that disconnects the curve into two pieces, one of genus i and the other of genus $g - i$.

The above classes satisfy the following important relations that we shall make use of:

- $\kappa = 12\lambda - \sum_i \delta_i$, called the **Mumford relation**; and
- $\pi_*(K \cdot \pi^*\lambda) = (2g-2)\lambda$ (note that this follows from the Push-Pull Formula, see part (b) of Proposition 44).

4.3.4 Weierstrass Divisors

In § 4.3.1, we computed (under certain generality assumptions) the number of type (a) and type (b) Weierstrass points in the fibers of an admissible family of curves X/B. In particular, for any such family, there is a divisor on the base B such that for every point in the support of the divisor, the corresponding fibers have type (a) or (b) Weierstrass points. Suppose now that the fibers of the family X/B have only nodal singularities. By making the replacements

$$(X \rightsquigarrow \overline{\mathcal{C}}_g, B \rightsquigarrow \overline{\mathcal{M}}_g),$$

we deduce that the locus of curves with a type (a) Weierstrass point forms a divisor on $\overline{\mathcal{M}}_g$, and so does the locus of curves with a type (b) Weierstrass point. Let these divisors be denoted $\overline{\mathcal{W}}^{(a)}$ and $\overline{\mathcal{W}}^{(b)}$, respectively; it is natural to ask what the Chow classes of these divisors are.

Fortunately, we can use the computations of the classes of type (a) and type (b) Weierstrass points in § 4.3.1 to begin to understand the Chow classes of $\overline{\mathcal{W}}^{(a)}$ and $\overline{\mathcal{W}}^{(b)}$. If $\widetilde{\delta}_i$ denotes the class of singular points in the fibers of X/B with the property that the containing fiber is a reducible union of two curves, one of genus i and the other of genus $g - i$, then all we need to do is make the replacements

$$(c_1(\omega_{X/B}) \rightsquigarrow K, c_1(\pi_*\omega_{X/B}) \rightsquigarrow \lambda, \widetilde{\delta}_i \rightsquigarrow \delta_i)$$

and apply the relations described in § 4.3.3 to simplify our answers.

Remark 111. Before we proceed, there is one caveat: the analysis in § 4.3.1 depends on the fact that the fibers of our family are irreducible, so we can only really solve for the λ and δ_0 terms in the Chow classes of $\overline{\mathcal{W}}^{(a)}$ and $\overline{\mathcal{W}}^{(b)}$. The issue is that when the singular fibers are reducible, the maps ϕ', ψ defined in § 4.3.1 degenerate along entire irreducible components of the singular fibers, so the degeneracy loci of these maps fail to have the codimension required for Porteous' Formula (Theorem 54) to apply. A version of Porteous' Formula exists for maps that have such "excess degeneracy" (indeed, see [Ful98, Example 14.4.7]), but it is quite a ways more complicated than the edition of the formula stated in Theorem 54. In [Ble12], this special version of Porteous' Formula was applied to compute the δ_1 term for the divisor of hyperellitpic curves in $\overline{\mathcal{M}}_3$, but the analysis therein is *ad hoc* and does not readily generalize to treating the case of arbitrary genus.

We are now in position to compute the (λ and δ_0 terms of the) divisor classes $[\overline{\mathcal{W}}^{(a)}]$ and $[\overline{\mathcal{W}}^{(b)}]$. Indeed, for the case of type (a) Weierstrass points, we have that

$$\begin{aligned}[\overline{\mathcal{W}}^{(a)}] &= \pi_* \left(\frac{1}{24} m(m-1)(m+1)(3m+2) \cdot K^2 - \frac{1}{2} m(m+1) \cdot K \cdot \pi^* \lambda - \binom{m+1}{4} \cdot \widetilde{\delta}_0 \right) \\ &= \frac{1}{24} m(m-1)(m+1)(3m+2) \cdot (12\lambda - \delta_0) - \frac{1}{2} m(m+1) \cdot (2g-2)\lambda - \binom{m+1}{4} \cdot \delta_0\end{aligned}$$

Substituting in the value of $m = \dim_k W + 1$ from Lemma 108, we obtain the following theorem.

Theorem 112. *We have that the class of type (a) Weierstrass points is given by*

$$[\overline{\mathcal{W}}^{(a)}] = \frac{1}{2}(g+1)(g+2)(3g^2+2g+2)\lambda - \frac{1}{6} g(1+g)^2(2+g) \cdot \delta_0$$

if $n = 1$, *and if* $n > 1$, *we have that*

$$\begin{aligned}[\overline{\mathcal{W}}^{(a)}] = \frac{1}{2}&(g-1)(2-g-2n+2gn)(3-g-2n+2gn) \cdot \\ &(-10+3g+22n-12gn-12n^2+12gn^2)\lambda - \\ &\frac{1}{24}(-1+g)(-1+2n)(2-g-2n+2gn) \cdot \\ &(3-g-2n+2gn)(8-3g-6n+6gn)\delta_0.\end{aligned}$$

Following the same line of reasoning for the case of type (b) Weierstrass points, we have that

$$[\overline{\mathcal{W}}^{(b)}] = \pi_* \left(\frac{1}{24} m(m-2)(m-1)(3m-5) \cdot K^2 - \frac{1}{2}(m-2)(m-1) \cdot K \cdot \pi^* \lambda + (\pi^* \lambda)^2 - \binom{m}{4} \cdot \widetilde{\delta}_0 \right)$$

$$= \frac{1}{24}m(m-2)(m-1)(3m-5)\cdot(12\lambda-\delta_0) - \frac{1}{2}(m-2)(m-1)\cdot(2g-2)\lambda - \binom{m}{4}\cdot\delta_0$$

Once more, upon substituting in the value of $m = \dim_k W + 1$ from Lemma 108, we obtain the following theorem.

Theorem 113. *We have that the class of type (b) Weierstrass points is given by*

$$[\overline{\mathcal{W}}^{(b)}] = \frac{1}{2}(-1+g)g^2(-1+3g)\lambda - \frac{1}{6}(-1+g)^2g(1+g)\delta_0$$

if $n = 1$*, and if* $n > 1$*, we have that*

$$\begin{aligned}[\overline{\mathcal{W}}^{(b)}] = \frac{1}{2}&(-1+g)(-1+2n)(-g-2n+2gn)\cdot \\ &(4-9g+3g^2-14n+26gn-12g^2n+12n^2-24gn^2+12g^2n^2)\lambda- \\ \frac{1}{6}&(-1+g)(-1+2n)(-g-2n+2gn)^2(2-g-2n+2gn)\delta_0.\end{aligned}$$

Remark 114. The divisor $[\overline{\mathcal{W}}^{(b)}]$ was first computed for $n = 1$ by S. Diaz in [Dia85]. Subsequently, F. Cukierman used an argument in [Cuk89] involving the Riemann-Hurwitz Formula to deduce the divisor $[\overline{\mathcal{W}}^{(a)}]$ from Diaz's result, still for $n = 1$. The only result on higher Weierstrass points that we are aware of is from [CF91], where the divisor of weight-1 Weierstrass points in $\overline{\mathcal{C}}_g$ is computed. Finally, we note that a special case of the method we described above was employed by Esteves in [Est16] to compute the class of the locus of hyperelliptic curves in $\overline{\mathcal{M}}_3$. The key advantage of our method of computing these Weierstrass divisors is that it is more general and more direct, requiring minimal use of *ad hoc* techniques that only work in a specific genus.

Bibliography

[B06́] É. Bézout. *General theory of algebraic equations*. Princeton University Press, Princeton, NJ, 2006. Translated from the 1779 French original by Eric Feron.

[Ble12] T. Bleier. Excess Porteous, coherent Porteous, and the hyperelliptic locus in $\overline{\mathcal{M}}_3$. *Michigan Math. J.*, 61(2):359–383, 2012.

[Cau06] S. Cautis. *Extending families of curves: Monodromy and applications*. PhD thesis, 2006. Thesis (Ph.D.)–Harvard University.

[Cay09] A. Cayley. *The collected mathematical papers. Volume 5*. Cambridge Library Collection. Cambridge University Press, Cambridge, 2009. Reprint of the 1892 original.

[CF91] F. Cukierman and L.-Y. Fong. On higher Weierstrass points. *Duke Math. J.*, 62(1):179–203, 1991.

[Cho56] W.-L. Chow. On equivalence classes of cycles in an algebraic variety. *Ann. of Math. (2)*, 64:450–479, 1956.

[Cuk89] F. Cukierman. Families of Weierstrass points. *Duke Math. J.*, 58(2):317–346, 1989.

[Dia85] S. Diaz. Exceptional Weierstrass points and the divisor on moduli space that they define. *Mem. Amer. Math. Soc.*, 56(327):iv+69, 1985.

[DM69] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.

[EH16] D. Eisenbud and J. Harris. *3264 and all that Intersection Theory in Algebraic Geometry*. Cambridge University Press, Cambridge, UK, 2016.

[Est96] E. Esteves. Wronski algebra systems on families of singular curves. *Ann. Sci. École Norm. Sup. (4)*, 29(1):107–134, 1996.

[Est16] E. Esteves. The stable hyperelliptic locus in genus 3: an application of Porteous formula. *J. Pure Appl. Algebra*, 220(2):845–856, 2016.

[Ful98] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[Gro67] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, (32):361, 1967.

[Har80] R. Hartshorne. Stable reflexive sheaves. *Math. Ann.*, 254(2):121–176, 1980.

[Har95] J. Harris. *Algebraic geometry*, volume 133 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. A first course, Corrected reprint of the 1992 original.

[HM98] J. Harris and I. Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[Kle85] S. Kleiman. Book Review: Intersection theory; Book Review: Introduction to intersection theory in algebraic geometry. *Bull. Amer. Math. Soc. (N.S.)*, 12(1):137–143, 1985.

[Lee12] H. Lee. A note on the Hilbert scheme of points on a cusp curve. *Internat. J. Algebra Comput.*, 22(3):1250025, 9, 2012.

[LT94] D. Laksov and A. Thorup. Weierstrass points and gap sequences for families of curves. *Ark. Mat.*, 32(2):393–422, 1994.

[Mor71] W. Morris, editor. *The American Heritage Dictionary of the English Language*. American Heritage, NY and Houghton Mifflin, MA, 1971.

[Mum83] D. Mumford. Towards an enumerative geometry of the moduli space of curves. In *Arithmetic and geometry, Vol. II*, volume 36 of *Progr. Math.*, pages 271–328. Birkhäuser Boston, Boston, MA, 1983.

[New72] I. Newton. *Philosophiae naturalis principia mathematica. Vol. I.* Harvard University Press, Cambridge, Mass., 1972. Reprinting of the third edition (1726) with variant readings, assembled and edited by Alexandre Koyré and I. Bernard Cohen with the assistance of Anne Whitman.

[PS85] R. Piene and M. Schlessinger. On the Hilbert scheme compactification of the space of twisted cubics. *Amer. J. Math.*, 107(4):761–774, 1985.

[Ran05a] Z. Ran. Cycle map on Hilbert schemes of nodal curves. In *Projective varieties with unexpected properties*, pages 361–378. Walter de Gruyter, Berlin, 2005.

[Ran05b] Z. Ran. Geometry on nodal curves. *Compos. Math.*, 141(5):1191–1212, 2005.

[Ran05c] Z. Ran. A note on Hilbert schemes of nodal curves. *J. Algebra*, 292(2):429–446, 2005.

[Ran13] Z. Ran. Tautological module and intersection theory on Hilbert schemes of nodal curves. *Asian J. Math.*, 17(2):193–263, 2013.

[Sal60] G. Salmon. *A treatise on the higher plane curves: intended as a sequel to "A treatise on conic sections"*. 3rd ed. Chelsea Publishing Co., New York, 1960.

[Vak17] R. Vakil. The Rising Sea: Foundations of Algebraic Geometry. http://math.stanford.edu/~vakil/216blog, 2017.

[WL90] C. Widland and R. Lax. Weierstrass points on Gorenstein curves. *Pacific J. Math.*, 142(1):197–208, 1990.