A1 Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) be a function such that \( f(x, y) + f(y, z) + f(z, x) = 0 \) for all real numbers \( x, y, \) and \( z \). Prove that there exists a function \( g : \mathbb{R} \rightarrow \mathbb{R} \) such that \( f(x, y) = g(x) - g(y) \) for all real numbers \( x \) and \( y \).

A2 Alan and Barbara play a game in which they take turns filling entries of an initially empty \( 2008 \times 2008 \) array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

A3 Start with a finite sequence \( a_1, a_2, \ldots, a_n \) of positive integers. If possible, choose two indices \( j < k \) such that \( a_j \) does not divide \( a_k \), and replace \( a_j \) and \( a_k \) by \( \gcd(a_j, a_k) \) and \( \operatorname{lcm}(a_j, a_k) \), respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (Note: \( \gcd \) means greatest common divisor and \( \operatorname{lcm} \) means least common multiple.)

A4 Define \( f : \mathbb{R} \rightarrow \mathbb{R} \) by

\[
  f(x) = \begin{cases} 
    x & \text{if } x \leq e \\
    xf(\ln x) & \text{if } x > e.
  \end{cases}
\]

Does \( \sum_{n=1}^{\infty} \frac{1}{f(n)} \) converge?

A5 Let \( n \geq 3 \) be an integer. Let \( f(x) \) and \( g(x) \) be polynomials with real coefficients such that the points \((f(1), g(1)), (f(2), g(2)), \ldots, (f(n), g(n))\) in \( \mathbb{R}^2 \) are the vertices of a regular \( n \)-gon in counterclockwise order. Prove that at least one of \( f(x) \) and \( g(x) \) has degree greater than or equal to \( n - 1 \).

A6 Prove that there exists a constant \( c > 0 \) such that in every nontrivial finite group \( G \) there exists a sequence of length at most \( c \ln |G| \) with the property that each element of \( G \) equals the product of some subsequence. (The elements of \( G \) in the sequence are not required to be distinct. A subsequence of a sequence is obtained by selecting some of the terms, not necessarily consecutive, without reordering them; for example, \( 4, 4, 2 \) is a subsequence of \( 2, 4, 6, 4, 2 \), but \( 2, 2, 4 \) is not.)

B1 What is the maximum number of rational points that can lie on a circle in \( \mathbb{R}^2 \) whose center is not a rational point? (A rational point is a point both of whose coordinates are rational numbers.)

B2 Let \( F_0(x) = \ln x \). For \( n \geq 0 \) and \( x > 0 \), let \( F_{n+1}(x) = \int_0^x F_n(t) \, dt \). Evaluate

\[
  \lim_{n \to \infty} \frac{n! F_n(1)}{\ln n}.
\]

B3 What is the largest possible radius of a circle contained in a 4-dimensional hypercube of side length 1?

B4 Let \( p \) be a prime number. Let \( h(x) \) be a polynomial with integer coefficients such that \( h(0), h(1), \ldots, h(p^2 - 1) \) are distinct modulo \( p^2 \). Show that \( h(0), h(1), \ldots, h(p^3 - 1) \) are distinct modulo \( p^3 \).

B5 Find all continuously differentiable functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that for every rational number \( q \), the number \( f(q) \) is rational and has the same denominator as \( q \). (The denominator of a rational number \( q \) is the unique positive integer \( b \) such that \( q = a/b \) for some integer \( a \) with \( \gcd(a, b) = 1 \).) (Note: \( \gcd \) means greatest common divisor.)

B6 Let \( n \) and \( k \) be positive integers. Say that a permutation \( \sigma \) of \( \{1, 2, \ldots, n\} \) is \( k \)-limited if \( |\sigma(i) - i| \leq k \) for all \( i \). Prove that the number of \( k \)-limited permutations of \( \{1, 2, \ldots, n\} \) is odd if and only if \( n \equiv 0 \) or 1 (mod \( 2k + 1 \)).