A1 Find the volume of the region of points \((x, y, z)\) such that
\[
(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).
\]

A2 Alice and Bob play a game in which they take turns removing stones from a heap that initially has \(n\) stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many \(n\) such that Bob has a winning strategy. (For example, if \(n = 17\), then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

A3 Let \(1, 2, 3, \ldots, 2005, 2006, 2007, 2009, 2012, 2016, \ldots\) be a sequence defined by \(x_k = k\) for \(k = 1, 2, \ldots, 2006\) and \(x_{k+1} = x_k + x_{k-2005}\) for \(k \geq 2006\). Show that the sequence has 2005 consecutive terms each divisible by 2006.

A4 Let \(S = \{1, 2, \ldots, n\}\) for some integer \(n > 1\). Say a permutation \(\pi\) of \(S\) has a local maximum at \(k \in S\) if

(i) \(\pi(k) > \pi(k+1)\) for \(k = 1\);

(ii) \(\pi(k-1) < \pi(k)\) and \(\pi(k) > \pi(k+1)\) for \(1 < k < n\);

(iii) \(\pi(k-1) < \pi(k)\) for \(k = n\).

(For example, if \(n = 5\) and \(\pi\) takes values at \(1, 2, 3, 4, 5\) of \(2, 1, 4, 5, 3\), then \(\pi\) has a local maximum of 2 at \(k = 1\), and a local maximum of 5 at \(k = 4\).) What is the average number of local maxima of a permutation of \(S\), averaging over all permutations of \(S\)?

A5 Let \(n\) be a positive odd integer and let \(\theta\) be a real number such that \(\theta/\pi\) is irrational. Set \(a_k = \tan(\theta + k\pi/n)\), \(k = 1, 2, \ldots, n\). Prove that

\[
a_1 + a_2 + \cdots + a_n \quad a_1a_2 \cdots a_n
\]

is an integer, and determine its value.

A6 Four points are chosen uniformly and independently at random in the interior of a given circle. Find the probability that they are the vertices of a convex quadrilateral.