Lecture 13
Ordinary differential equations.

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1. Linear equations with constant coefficients.

2. Variation of constants.
   - The parametrix expansion.
Today I want to discuss some facts about ordinary differential equations. A course on dynamical systems given 40 years ago would consist almost entirely in the study of ordinary differential equations. We are only going to devote six or so lectures to this subject.

Today’s lecture will be some mix of slides and demonstrations using pplane.
We have proved the local existence theorem for equations of the form

\[ x'(t) = F(t, x), \quad x(0) = x_0 \]

under Lipschitz assumptions on \( F \) via the contraction fixed point theorem. I first want to concentrate on equations where \( F \) does not depend on \( t \).
Outline
Linear equations with constant coefficients.

Variation of constants.

Linear homogenous equations with constant coefficients.

These are equations of the form

\[ x' = Ax \]

where \( A \) is a constant bounded linear operator on a Banach space, \( X \). You may as well think of \( X \) as a finite dimensional vector space. We must also specify the initial conditions, of course.

In case \( X = \mathbb{R} \), so \( A = a \) is a scalar, we know that the general solution to the above equation is \( x(t) = ce^{at} \) where the constant \( c \) is determined by the initial condition, \( c = x(0) \) and

\[
e^{at} = 1 + at + \frac{1}{2} a^2 t^2 + \frac{1}{3!} a^3 t^3 + \cdots .
\]

Exactly the same method works in general!
The exponential series for an operator.

Define

\[ e^{tA} = I + tA + \frac{1}{2} t^2 A^2 + \frac{1}{3!} t^3 A^3 + \cdots. \]

Here \( I \) is the identity operator so \( e^{0A} = I \) is the identity. The exact same proof of the convergence of the exponential series in one variable shows that this series converges for all \( t \) and that

\[ e^{(s+t)A} = e^{sA} e^{tA}. \]

It is \textit{not true} in general that \( e^{tA} e^{tB} = e^{t(A+B)} \). This lack of equality stems from the fact that \( A \) and \( B \) may not commute, and this fact is manifested in the physical world by quantum mechanics. But if \( A \) and \( B \) do commute then \( e^{tA} e^{tB} = e^{t(A+B)} \).
The derivative of the exponential.

We may differentiate the exponential series with respect to $t$ term by term and we find that

$$\frac{d}{dt} e^{tA} = Ae^{tA} = e^{tA}A.$$

In particular, if we set

$$x(t) = e^{tA}x_0$$

then

$$x'(t) = Ax(t) \quad \text{and} \quad x(0) = x_0.$$

So the study of linear homogeneous differential equations with constant coefficients reduces to the analysis of $e^{tA}$. 
\[ e^{tPAP^{-1}} = Pe^{tA}P^{-1}. \]

Suppose that \( P \) is an invertible operator and

\[ B = PAP^{-1}. \]

then \( B^2 = PA^2P^{-1}, \ B^3 = PA^3P^{-1} \) etc. so

\[ e^{tB} = Pe^{tA}P^{-1}. \]

We know from linear algebra that (in finite dimensions) every \( B \) is of the form \( B = PAP^{-1} \) where \( A \) has a “nice” normal form. So our study is reduced to understanding \( e^{tA} \) for normal forms (and also understanding the effect of conjugating by \( P \)).

Here is a complete analysis in two (real) dimensions:
\( B \) has two real distinct eigenvalues.

Then \( B \) can be diagonalized, i.e. \( B = PAP^{-1} \) where

\[
A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}
\]

and

\[
e^{tA} = \begin{pmatrix} e^{ta} & 0 \\ 0 & e^{tb} \end{pmatrix}.
\]

The \( x \) and \( y \) axes are invariant. If \( a < 0 \) and \( b < 0 \) then \( e^{tA} \) for \( t > 0 \) contracts all points at different rates toward the origin. The effect of the conjugation by \( P \) is to replace the \( x \) and \( y \) axes by other lines through the origin.
If $a > 0$ and $b < 0$ then $e^{tA}$ is hyperbolic for $t \neq 0$, for $t > 0$ expanding in the $x$-direction and contracting in the $y$-direction. If $a > 0$ and $b > 0$, then $e^{tA}$ expands for positive $t$. If $a = 0$ and $b < 0$ then points on the $x$-axis are stationary and points all converge toward the $x$-axis. Similar analysis for $a > 0$ and $b = 0$. 
Two equal real eigenvalues.

Here there are two cases: if

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

then

$$e^{tA} = e^{ta}I.$$

So uniform expansion if $a > 0$, uniform contraction if $a < 0$ and $e^{tA} \equiv I$ if $A = 0$. Since an $A$ of this form commutes with all other matrices, if $B = PAP^{-1}$ then $B = A$. 
The second case is

\[ A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}. \]

This is of the form \( A = aI + C \) where

\[ C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]

so

\[ e^{tA} = e^{ta} e^{tC}. \]

To compute \( e^{tC} \) observe that \( C^2 = 0 \) so

\[ e^{tC} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \]
Non-real eigenvalues.

If the eigenvalues are \( a \pm ib \) with \( b \neq 0 \) then a normal form is

\[
A = \begin{pmatrix} a & b \\ -b & 0 \end{pmatrix} = aI + b \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

So if

\[
C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

then

\[
e^{tA} = e^{ta}e^{tbC}
\]

so we must compute \( e^{tC} \).
This follows from

\[ C^2 = -I \quad \text{so} \quad C^3 = -C, \quad C^4 = I. \]

Thus

\[ e^{tC} = \begin{pmatrix} 1 - \frac{1}{2} t^2 + \frac{1}{4!} t^4 + \cdots & t - \frac{1}{3!} t^3 + \cdots \\ -t + \frac{1}{3!} t^3 - \cdots & 1 - \frac{1}{2} t^2 + \frac{1}{4!} t^4 + \cdots \end{pmatrix}. \]
Thus if \( a = 0 \), the trajectories of \( e^{tA} \) are circles with velocity of rotation \( b \). If \( a < 0 \) the trajectories are circular spirals heading into the origin as \( t \to \infty \), and if \( a > 0 \) the trajectories are circular spirals spiraling out.

The effect of conjugating by \( P \) is to replace the circles by ellipses.
If no eigenvalue has real part zero, we have the “infinitesimal version” of hyperbolicity, and an old differential equation theorem of mine says that if there is a zero of the vector field (say at 0) and the matrix of partial derivatives at the zero is hyperbolic in the above sense, then the behavior of the non-linear system near enough to the fixed point is equivalent, via a homoemorphism, to the linearized equation.
But, just as we studied in the behavior of maps in one dimension near fixed points with $f'(p) = \pm 1$ there is a similar study of bifurcations of non-hyperbolic zeros of vector fields. A famous example is the *Hopf bifurcation*. Here is an illustration of this phenomenon, without going into the technical details. Suppose our system of differential equations is

\[
\begin{align*}
  x' &= ax + y - x(x^2 + y^2) \\
  y' &= ay - x - y(x^2 + y^2).
\end{align*}
\]
If $a < 0$, all trajectories spiral into the origin. If $a > 0$ and small, then the origin has become an unstable fixed point, and all trajectories starting near the origin spiral outward towards a periodic trajectory while points outside the periodic trajectory spiral in towards it.

In other words, as $a$ passes from negative to positive, an attractive fixed point has become repulsive and a nearby attractive periodic orbit has appeared.
So far we have been studying the homogeneous equation

\[ x'(t) = Ax(t), \quad x(0) = x_0. \]

Suppose we want to solve the "inhomogeneous" equation

\[ \frac{d}{dt}x(t) = Hx(t) + f(t) \]

where \( f \) is given, and with the initial condition \( x(0) = x_0 \). The solution is given by Lagrange's variation of constants formula

\[ x(t) = e^{tH}x_0 + \int_0^t e^{(t-s)H}f(s)\,ds \]

as can be checked by differentiating the right hand side. This formula is also known as Duhamel's formula.
If $F$ is an operator valued function of $t$ then the operator version of the above says that

$$X(t) = e^{tH} + \int_0^t e^{(t-s)H} F(s) ds \quad (1)$$

is the solution to the differential equation

$$\frac{d}{dt}X = HX + F$$

with the initial conditions

$$X(0) = I.$$
For example, suppose that $H = H_0 + H_1$ and we take $F(t) = H_1 e^{tH}$. We want to find a solution to

$$
\frac{d}{dt} Y(t) = HY(t) = H_0 Y(t) + H_1 Y(t), \quad Y(0) = I.
$$

The variation of constants formula tell us that

$$
Y(t) = e^{tH_0} + \int_0^t e^{(t-s)H_0} H_1 e^{sH} ds. \quad (2)
$$

If we substitute this formula into itself (i.e. use this formula for the $e^{sH}$ occurring in the integral on the right) we get

$$
e^{tH} = e^{tH_0} + \int_0^t e^{(t-s)H_0} H_1 e^{sH_0} dt + \int_0^t \int_0^s e^{(t-s)H_0} H_1 e^{(s-\tau)H_0} H_1 e^{\tau H} d\tau ds
$$
Clearly we can keep going. The usefulness of this scheme is as follows. Suppose that $H_1$ is small, for example suppose that we can ignore all terms involving three products of $H_1$. Then in the above expression, replacing $e^{\tau H}$ by $e^{\tau H_0}$ on the right, we get an approximate expression for $e^{tH}$ in terms of integrals involving $e^{tH_0}$ and products of $H_1$. The corresponding series (and approximation) is known as the Volterra series (or approximations) to mathematicians, and is known as the Born series (or approximations) to physicists.
Suppose we want to find $e^{tH}$ and we only found an approximate solution - we have found an operator valued function $K(t)$ such that

$$\frac{dK(t)}{dt} = HK(t) + R(t), \quad K(0) = I.$$ 

The variation of constants formula tells us that

$$K(t) = e^{tH} + \int_0^t e^{(t-s)H} R(s) ds$$

which we shall write as

$$e^{tH} = K(t) - \int_0^t e^{(t-s)H} R(s) ds.$$ 

Substitute this back into itself to obtain

$$e^{tH} = K(t) - \int_0^t K(t-s)R(s) ds + \int_0^t \int_0^{t-s} e^{(t-s-\tau)H} R(\tau) R(s) d\tau ds.$$ 

Keep going. This suggests the following:
Let \( \Delta_k \) denote the \( k \)-simplex

\[
\Delta_k = \{(t_1, \ldots, t_k | 0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq 1}\}.
\]

If all the \( t_i \) are unequal, there are \( k! \) ways of reordering them. Since we may ignore possible equalities in computing volume, this shows that the Euclidean volume of \( \Delta_k \) is \( 1/k! \).

So the volume of \( t\Delta_k \) is \( t^k/k! \). Define the operators \( Q(k, t) \) by

\[
Q(k, t) := \int_{t\Delta_k} K(t - t_k)R(t_k - t_{k-1}) \cdots R(t)dt_1 \cdots dt_k.
\]

So this integral is over \( 0 \leq t_1 \leq t_2 \cdots \leq t_k \leq t \). To shorten the formulas, I will drop the \( dt_1 \cdots dt_k \).
If $K$ and $R$ are uniformly bounded, say by $C$, in the interval $[0, T]$, then the $Q(k, t)$ are bounded by $C^k t^k / k!$ and so the series

$$
\sum_{k=0}^{\infty} (-1)^k Q(k, t)
$$

converges uniformly. $R(k, s)$ by

$$
R(k, s) = \int_{s\Delta_{k-1}}^{s} R(s - t_{k-1})rR(t_{k-1} - t_{k-2}) \cdots R(t_2 - t_1)R(t_1)
$$

so that

$$
Q(k, t) = \int_{0}^{t} K(t - s)R(k, s)ds.
$$
The parametrix expansion.

\[ Q(k, t) = \int_0^t K(t - s)R(k, s)ds. \]

If we apply \( \left( \frac{d}{dt} - H \right) \) to \( Q(k, t) \) we get

\[ R(k, s) + R(k + 1, s) \]

so the sum in

\[ \left[ \frac{d}{dt} - H \right] \left( \sum_{k=0}^{\infty} (-1)^k Q(k, t) \right) \]

telescopes to zero. Hence

\[ \sum_{k=0}^{\infty} (-1)^k Q(k, t) \]

is a solution to our search for \( e^{tH} \) starting with an approximate solution. This method is known as the parametrix expansion.