Lecture 3
The logistic family.

Shlomo Sternberg
1 The logistic family.
   - The scenario for $0 < \mu \leq 1$.
   - The scenario for $1 < \mu \leq 3$.
   - Period doubling bifurcations in the logistic family.

2 The period doubling bifurcation.
In population biology one considers iteration of the "logistic function"

\[ L_\mu(x) := \mu x(1 - x). \] (1)

Here \(0 < \mu\) is a real parameter.

The fixed points of \(L_\mu\) are 0 and \(1 - \frac{1}{\mu}\). Since \(L'_\mu(x) = \mu - 2\mu x\),

\[ L'_\mu(0) = \mu, \quad L'_\mu\left(1 - \frac{1}{\mu}\right) = 2 - \mu. \]

As \(x\) represents a proportion of a population, we are mainly interested only in \(0 \leq x \leq 1\). The maximum of \(L_\mu\) is always achieved at \(x = \frac{1}{2}\), and the maximum value is \(\frac{\mu}{4}\). So for \(0 < \mu \leq 4\), \(L_\mu\) maps \([0, 1]\) into itself.
Wht happens when $\mu > 4$?

For $\mu > 4$, portions of $[0, 1]$ are mapped into the range $x > 1$. A second operation of $L_\mu$ maps these points to the range $x < 0$ and then are swept off to $-\infty$ under successive applications of $L_\mu$.

We now examine the behavior of $L_\mu$ more closely for varying ranges of $\mu$ as $0 \leq \mu \leq 4$. 
For $0 < \mu < 1$, 0 is the only fixed point of $L_\mu$ on $[0,1]$ since the other fixed point, $1 - \frac{1}{\mu}$, is negative.
0 is an attractive fixed point for $0 < \mu < 1$.

For $0 < \mu < 1$, 0 is the only fixed point of $L_\mu$ on $[0, 1]$ since the other fixed point, $1 - \frac{1}{\mu}$, is negative. On this range of $\mu$, the point 0 is an attracting fixed point since $0 < L'_\mu(0) = \mu < 1$. Under iteration, all points of $[0, 1]$ tend to 0 under the iteration. The population “dies out”. 
The scenario for $0 < \mu \leq 1$.

$\mu = 1$.

For $\mu = 1$ we have

$$L_1(x) = x(1 - x) < x, \quad \forall x > 0.$$ 

Each successive application of $L_1$ to an $x \in (0, 1]$ decreases its value. The limit of the successive iterates cannot be positive since 0 is the only fixed point. So all points in $(0, 1]$ tend to 0 under iteration, but ever so slowly, since $L_1'(0) = 1$. In fact, for $x < 0$, the iterates drift off to more negative values and then tend to $-\infty$. 
For all $\mu > 1$, the fixed point, 0, is repelling, and the unique other fixed point, $1 - \frac{1}{\mu}$, lies in $[0, 1]$. 
The scenario for $1 < \mu \leq 3$.

$1 < \mu < 3$.

For $1 < \mu < 3$ we have

$$|L'_\mu(1 - \frac{1}{\mu})| = |2 - \mu| < 1,$$

so the non-zero fixed point is attractive.

We will see that the basin of attraction of $1 - \frac{1}{\mu}$ is the entire open interval $(0, 1)$, but the behavior is slightly different for the two domains, $1 < \mu \leq 2$ and $2 < \mu < 3$:

In the first of these ranges there is a steady approach toward the fixed point from one side or the other; in the second, the iterates bounce back and forth from one side to the other as they converge in towards the fixed point. The graphical iteration spirals in. Here are the details:
The scenario for $1 < \mu \leq 3$.

$1 < \mu \leq 2$.

For $1 < \mu < 2$ the non-zero fixed point lies between $0$ and $\frac{1}{2}$ and the derivative at this fixed point is $2 - \mu$ and so lies between $1$ and $0$. Here is the graph for $\mu = 1.5$: 

![Graph for $\mu = 1.5$]
Suppose that $x$ lies between 0 and the fixed point, $1 - \frac{1}{\mu}$. For this range of $x$ we have

$$\frac{1}{\mu} < 1 - x$$

so, multiplying by $\mu x$ we get

$$x < \mu x (1 - x) = L_{\mu}(x).$$

Also, $L_{\mu}$ is monotone increasing for $0 < x < \frac{1}{2}$. So for $x < 1 - \frac{1}{\mu}$, $L_{\mu}(x) < L_{\mu}(1 - \frac{1}{\mu}) = 1 - \frac{1}{\mu}$. Thus the iterates steadily increase toward $1 - \frac{1}{\mu}$, eventually converging geometrically with a rate close to $2 - \mu$. 

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If

$$1 - \frac{1}{\mu} < x$$

then $1 - x < \frac{1}{\mu}$ so, multiplying by $\mu x$ gives

$$L_\mu(x) < x.$$
If, in addition,
\[ x \leq \frac{1}{\mu} \]
then
\[ L_\mu(x) \geq 1 - \frac{1}{\mu}. \]

To see this observe that the function \( L_\mu \) has only one critical point, and that is a maximum. Since \( L_\mu(1 - \frac{1}{\mu}) = L_\mu(\frac{1}{\mu}) = 1 - \frac{1}{\mu} \), we conclude that the minimum value is achieved at the end points of the interval \([1 - \frac{1}{\mu}, \frac{1}{\mu}]\).
Finally, for

$$\frac{1}{\mu} < x \leq 1, \quad L_\mu(x) < 1 - \frac{1}{\mu}.$$  

So on the range $1 < \mu < 2$ the behavior of $L_\mu$ is as follows: All points $0 < x < 1 - \frac{1}{\mu}$ steadily increase toward the fixed point, $1 - \frac{1}{\mu}$. All points satisfying $1 - \frac{1}{\mu} < x < \frac{1}{\mu}$ steadily decrease toward the fixed point. The point $\frac{1}{\mu}$ satisfies $L_\mu(\frac{1}{\mu}) = 1 - \frac{1}{\mu}$ and so lands on the non-zero fixed point after one application. The points satisfying $\frac{1}{\mu} < x < 1$ get mapped by $L_\mu$ into the interval $0 < x < 1 - \frac{1}{\mu}$, In other words, they overshoot the mark, but then steadily increase towards the non-zero fixed point. Of course $L_\mu(1) = 0$ which is always true.
$\mu = 2$ - the fixed point is superattractive.

When $\mu = 2$, the points $\frac{1}{\mu}$ and $1 - \frac{1}{\mu}$ coincide and equal $\frac{1}{2}$ with $L'_2\left(\frac{1}{2}\right) = 0$. There is no "steadily decreasing" region, and the fixed point, $\frac{1}{2}$ is superattractive - the iterates zoom into the fixed point faster than any geometrical rate.
The logistic family. The period doubling bifurcation.

The scenario for $1 < \mu \leq 3$. 

$2 < \mu < 3$. 

Here the fixed point $1 - \frac{1}{\mu} > \frac{1}{2}$ while $\frac{1}{\mu} < \frac{1}{2}$. The derivative at this fixed point is negative:

$$L'_\mu (1 - \frac{1}{\mu}) = 2 - \mu < 0.$$ 

So the fixed point $1 - \frac{1}{\mu}$ is an attractor, but as the iterates converge to the fixed points, they oscillate about it, alternating from one side to the other. The entire interval $(0, 1)$ is in the basin of attraction of the fixed point. To see this, will take some work.

Before going into the details of the argument, we illustrate the result via graphical iteration with $\mu = 2.5$ and initial point $x_0 = .75$. 

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The logistic family. The period doubling bifurcation.

The scenario for $1 < \mu \leq 3$.

Graphical iteration for $\mu = 2.5$, $x_0 = .75$. 

![Graphical iteration](image-url)
The scenario for $1 < \mu \leq 3$.

Proof that all of $(0, 1)$ is in the basin of attraction of $1 - \frac{1}{\mu}$.

The graph of $L_\mu$ lies entirely above the line $y = x$ on the interval $(0, 1 - \frac{1}{\mu}]$. In particular, it lies above the line $y = x$ on the subinterval $[\frac{1}{\mu}, 1 - \frac{1}{\mu}]$ and takes its maximum at $\frac{1}{2}$. So

$$\frac{\mu}{4} = L_\mu(\frac{1}{2}) > L_\mu(1 - \frac{1}{\mu}) = 1 - \frac{1}{\mu}.$$ Hence $L_\mu$ maps the interval $[\frac{1}{\mu}, 1 - \frac{1}{\mu}]$ onto the interval $[1 - \frac{1}{\mu}, \frac{\mu}{4}]$. 

The scenario for $1 < \mu \leq 3$.

$L_\mu$ maps the interval $\left[\frac{1}{\mu}, 1 - \frac{1}{\mu}\right]$ onto the interval $\left[1 - \frac{1}{\mu}, \frac{\mu}{4}\right]$. The map $L_\mu$ is decreasing to the right of $\frac{1}{2}$, so it is certainly decreasing to the right of $1 - \frac{1}{\mu}$. Hence it maps the interval $\left[1 - \frac{1}{\mu}, \frac{\mu}{4}\right]$ into an interval whose right hand end point is $1 - \frac{1}{\mu}$ and whose left hand end point is $L_\mu\left(\frac{\mu}{4}\right)$. We claim that

$$L_\mu\left(\frac{\mu}{4}\right) > \frac{1}{2}.$$ 

This amounts to showing that

$$\frac{\mu^2(4 - \mu)}{16} > \frac{1}{2}$$

or that

$$\mu^2(4 - \mu) > 8.$$
To show that 

$$\mu^2(4 - \mu) > 8 :$$

The critical points of $\mu^2(4 - \mu)$ are 0 and $\frac{8}{3}$ and the second derivative at $\frac{8}{3}$ is negative, so it is a local maximum. So we need only check the values of $\mu^2(4 - \mu)$ at the end points, 2 and 3, of the range of $\mu$ we are considering, where the values are 8 and 9.
The scenario for $1 < \mu \leq 3$.

The image of $\left[ \frac{1}{\mu}, 1 - \frac{1}{\mu} \right]$ is the same as the image of $\left[ \frac{1}{2}, 1 - \frac{1}{\mu} \right]$ and is $\left[ 1 - \frac{1}{\mu}, \frac{\mu}{4} \right]$. The image of this interval is the interval $\left[ L_\mu \left( \frac{\mu}{4} \right), 1 - \frac{1}{\mu} \right]$, with $\frac{1}{2} < L_\mu \left( \frac{\mu}{4} \right)$. If we apply $L_\mu$ to this interval, we get an interval to the right of $1 - \frac{1}{\mu}$ with right end point $L_\mu^2 \left( \frac{\mu}{4} \right) < L_\mu \left( \frac{1}{2} \right) = \frac{\mu}{4}$. The image of the interval $\left[ 1 - \frac{1}{\mu}, L_\mu^2 \left( \frac{\mu}{4} \right) \right]$ must be strictly contained in the image of the interval $\left[ 1 - \frac{1}{\mu}, \frac{\mu}{4} \right]$, and hence we conclude that

$$L_\mu^3 \left( \frac{\mu}{4} \right) > L_\mu \left( \frac{\mu}{4} \right).$$

Continuing in this way we see that under even powers, the image of $\left[ \frac{1}{2}, 1 - \frac{1}{\mu} \right]$ is a sequence of nested intervals whose right hand end point is $1 - \frac{1}{\mu}$ and whose left hand end points are

$$\frac{1}{2} < L_\mu \left( \frac{\mu}{4} \right) < L_\mu^3 \left( \frac{\mu}{4} \right) < \ldots .$$
The scenario for $1 < \mu \leq 3$.

\[
\frac{1}{2} < L_\mu \left( \frac{\mu}{4} \right) < L_\mu^3 \left( \frac{\mu}{4} \right) < \cdots .
\]

We claim that this sequence of points converges to the fixed point, $1 - \frac{1}{\mu}$. If not, it would have to converge to a fixed point of $L_\mu^2$ different from 0 and $1 - \frac{1}{\mu}$. We shall show that there are no such points. Indeed, a fixed point of $L_\mu^2$ is a zero of

\[
L_\mu^2(x) - x = \mu L_\mu(x)(1 - L_\mu(x)) = \mu [\mu x(1 - x)][1 - \mu x(1 - x)] - x.
\]

Two roots of this quartic polynomial, are the fixed points of $L_\mu$, which are 0 and $1 - \frac{1}{\mu}$. So the quartic polynomial factors into a quadratic polynomial times $\mu x(x - 1 + \frac{1}{\mu})$. A check shows that this quadratic polynomial is $-\mu^2 x^2 + (\mu^2 + \mu)x - \mu - 1$. The $b^2 - 4ac$ for this quadratic function is $\mu^2 (\mu^2 - 2\mu - 3) = \mu^2 (\mu + 1)(\mu - 3)$ which is negative for $2 < \mu < 3$ so the quadratic has no real roots.
We thus conclude that the iterates of any point in \((\frac{1}{\mu}, \frac{\mu}{4}]\) oscillate about the fixed point, \(1 - \frac{1}{\mu}\) and converge in towards it, eventually with the geometric rate of convergence a bit less than \(\mu - 2\). The graph of \(L_\mu\) is strictly above the line \(y = x\) on the interval \((0, \frac{1}{\mu}]\) and hence the iterates of \(L_\mu\) are strictly increasing so long as they remain in this interval. Furthermore they can’t stay there, for this would imply the existence of a fixed point in the interval and we know that there is none. Thus they eventually get mapped into the interval \([\frac{1}{\mu}, 1 - \frac{1}{\mu}]\) and the oscillatory convergence takes over.

Finally, since \(L_\mu\) is decreasing on \([1 - \frac{1}{\mu}, 1]\), any point in \([1 - \frac{1}{\mu}, 1)\) is mapped into \((0, 1 - \frac{1}{\mu}]\) and so converges to the non-zero fixed point.

In short, every point in \((0, 1)\) is in the basin of attraction of the non-zero fixed point and (except for the points \(\frac{1}{\mu}\) and the fixed point itself) eventually converge toward it in a “spiral” fashion.
Much of the analysis of the preceding case applies here. The differences are: the quadratic equation

\[-\mu^2 x^2 + (\mu^2 + \mu)x - \mu - 1.\]

for seeking points of period two now has a (double) root. But this root is \(\frac{2}{3} = 1 - \frac{1}{\mu}\) which is the fixed point. So there is still no point of period two other than the fixed points. The iterates continue to spiral in, but now ever so slowly since \(L'_\mu(\frac{2}{3}) = -1\).
\( \mu > 3 \), points of period two appear.

For \( \mu > 3 \) we have

\[
L_\mu'(1 - \frac{1}{\mu}) = 2 - \mu < -1
\]

so both fixed points, 0 and \( 1 - \frac{1}{\mu} \) are repelling. But now \(-\mu^2x^2 + (\mu^2 + \mu)x - \mu - 1.\) has two real roots which are

\[
p_{2\pm} = \frac{1}{2} + \frac{1}{2\mu} \pm \frac{1}{2\mu} \sqrt{(\mu + 1)(\mu - 3)}.
\]
\( \mu = 3.3 \), graphs of \( y = x, y = L_\mu(x), y = L_\mu^2(x) \).

Notice that now the graph of \( L_\mu^2 \) has four points of intersection with the line \( y = x \): the two (repelling) fixed points of \( L_\mu \) and two
The derivative of $L_\mu^2$ at these points of period two is given by

$$(L_\mu^2)'(p_{2\pm}) = L_\mu'(p_{2+})L_\mu'(p_{2-})$$

$$= (\mu - 2\mu p_{2+})(\mu - 2\mu p_{2-})$$

$$= \mu^2 - 2\mu^2(p_{2+} + p_{2-}) + 4\mu^2 p_{2+}p_{2-}$$

$$= \mu^2 - 2\mu^2(1 + \frac{1}{\mu}) + 4\mu^2 \times \frac{1}{\mu^2}(\mu + 1)$$

$$= -\mu^2 + 2\mu + 4.$$
Graphical iteration for $\mu = 3.3$, nine steps.

Notice the “spiraling out” from the fixed point.
Graphical iteration for $\mu = 3.3$, twenty five steps.

The attractive period two points become evident.
In this range the fixed points are repelling and both period two points are attracting. There will be points whose images end up, after a finite number of iterations, on the non-zero fixed point. All other points in $(0, 1)$ are attracted to the period two cycle. We omit the proof.
Superattracting period two points.

Notice also that there is a unique value of \( \mu \) in this range where

\[
p_{2+}(\mu) = \frac{1}{2}.
\]

Indeed, looking at the formula for \( p_{2+} \) we see that this amounts to the condition that \( \sqrt{(\mu + 1)(\mu - 3)} = 1 \) or

\[
\mu^2 - 2\mu - 4 = 0.
\]

The positive solution to this equation is given by \( \mu = s_2 \) where

\[
s_2 = 1 + \sqrt{5}.
\]

At \( s_2 \), the period two points are superattracting, since one of them coincides with \( \frac{1}{2} \) which is the maximum of \( L_{s_2} \).
Once $\mu$ passes $1 + \sqrt{6} = 3.449499...$ the points of period two become unstable and (stable) points of period four appear. Initially these are stable, but as $\mu$ increases they become unstable (at the value $\mu = 3.544090...$) and bifurcate into period eight points, initially stable.
Graphical iteration for $\mu = 3.46$, twenty five steps.

The attractive period four points become evident.
The total scenario so far, as $\mu$ increases from 0 to about 3.55, is as follows: For $\mu < b_1 := 1$, there is no non-zero fixed point. Past the first bifurcation point, $b_1 = 1$, the non-zero fixed point has appeared close to zero. When $\mu$ reaches the first superattractive value, $s_1 := 2$, the fixed point is at .5 and is superattractive. As $\mu$ increases, the fixed point continues to move to the right. Just after the second bifurcation point, $b_2 := 3$, the fixed point has become unstable and two stable points of period two appear, one to the right and one to the left of .5. The leftmost period two point moves to the right as we increase $\mu$, and at $\mu = s_2 := 1 + \sqrt{5} = 3.23606797...$ the point .5 is a period two point, and so the period two points are superattractive. When $\mu$ passes the second bifurcation value $b_2 = 1 + \sqrt{6} = 3.449...$ the period two points have become repelling and attracting period four points appear.
In fact, this scenario continues. The period $2^{n-1}$ points appear at bifurcation values $b_n$. They are initially attracting, and become superattracting at $s_n > b_n$ and become unstable past the next bifurcation value $b_{n+1} > s_n$ when the period $2^n$ points appear. The (numerically computed) bifurcation points and superstable points are tabulated as:
Period doubling bifurcations in the logistic family.

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<th>$b_n$</th>
<th>$s_n$</th>
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</tr>
<tr>
<td>$\infty$</td>
<td>3.569946</td>
<td>3.569946</td>
</tr>
</tbody>
</table>

The values of the $b_n$ are obtained by numerical experiment. Later, we shall describe a method for computing the $s_n$ using Newton’s method.
We should point out that this is still just the beginning of the story. For example, an attractive period three cycle appears at about 3.83. We shall come back to all of these points, but first go back and discuss theoretical problems associated to bifurcations, in particular, the period doubling bifurcation.
Review.

In the last lecture we studied the fold bifurcation, when the parameter $\mu$ where there is no (local) fixed point on one side of the bifurcation value, $b$, where a fixed point $p$ appears at $\mu = b$ with $F'_\mu(p) = 1$, and at the other side of $b$ the map $F_\mu$ has two fixed points, one attracting and the other repelling.

In order not to clutter up the notation, we assumed that coordinates have been chosen so that $b = 0$ and $p = 0$. So we make the standing assumption that $p = 0$ is a fixed point at $\mu = 0$, i.e. that

$$F(0, 0) = 0.$$

We proved:
The fold bifurcation.

Theorem

(Fold bifurcation). Suppose that at the point \((0, 0)\) we have

\[
\begin{align*}
(a) \quad & \frac{\partial F}{\partial x}(0, 0) = 1, \\
(b) \quad & \frac{\partial^2 F}{\partial x^2}(0, 0) > 0, \\
(c) \quad & \frac{\partial F}{\partial \mu}(0, 0) > 0.
\end{align*}
\]

Then there are non-empty intervals \((\mu_1, 0)\) and \((0, \mu_2)\) and \(\epsilon > 0\) so that

(i) If \(\mu \in (\mu_1, 0)\) then \(F_\mu\) has two fixed points in \((-\epsilon, \epsilon)\). One is attracting and the other repelling.

(ii) If \(\mu \in (0, \mu_2)\) then \(F_\mu\) has no fixed points in \((-\epsilon, \epsilon)\).
In the bifurcations we have seen in the logistic family, at the bifurcations values we have been studying the value of $F'_\mu(p) = -1$ at the fixed point $p$. The attractive fixed point becomes repelling and two attractive double points appear, as illustrated in the following diagram:
The logistic family.

The period doubling bifurcation.

attracting fixed point
repelling fixed point
attracting double point

\( \mu \)
The first period doubling bifurcation in the logistic family

To visualize the phenomenon we plot the function $L_\mu^2$ for the values $\mu = 2.9$ and $\mu = 3.3$. For $\mu = 2.9$ the curve crosses the diagonal at a single point, which is in fact a fixed point of $L_\mu$ and hence of $L_\mu^2$. This fixed point is stable. For $\mu = 3.3$ there are three crossings. The non-zero fixed point of $L_\mu$ has derivative smaller than $-1$, and hence the corresponding fixed point of $L_\mu^2$ has derivative greater than one. The two other crossings correspond to the stable period two orbit.
Lecture 3: The logistic family.
We now turn to the general theory: We are now assuming that \( \mu = 0 \) has 0 as a fixed point with \( F'_0(0) = -1 \). So the partial derivative of \( F(x, \mu) - x \) with respect to \( x \) is \(-2\) at \((0, 0)\). In particular it does not vanish, so we can now locally solve for \( x \) as a function of \( \mu \); there is (locally) a unique branch of fixed points, \( x(\mu) \), passing through the origin.

Let \( \lambda(\mu) \) denote the derivative of \( F_\mu \) with respect to \( x \) at the fixed point, \( x(\mu) \), i.e. define

\[
\lambda(\mu) := \frac{\partial F}{\partial x}(x(\mu), \mu).
\]
As notation, let us set

\[ F_\mu^2 := F_\mu \circ F_\mu \]

and define

\[ F^2(x, \mu) := F_\mu^2(x). \]

Notice that

\[ (F_\mu^2)'(x) = F_\mu'(F_\mu(x))F_\mu'(x) \]

by the chain rule so

\[ (F_0^2)'(0) = (F_0'(0))^2 = 1. \]

Hence

\[ (F_\mu^2)''(x) = F_\mu''(F_\mu(x))F_\mu'(x)^2 + F_\mu'(F_\mu(x))F_\mu''(x) \quad (2) \]

which vanishes at \( x = 0, \mu = 0 \). In other words,

\[ \frac{\partial^2 F^2}{\partial x^2}(0, 0) = 0. \quad (3) \]
\[ \frac{\partial^2 F^2}{\partial x^2}(0, 0) = 0. \quad (3) \]

Let us absorb the import of this equation. One might think that if we set \( G_\mu = F^2_\mu \), then \( G'_\mu(0) = 1 \), so all we need to do is apply The fold bifurcation theorem to \( G_\mu \). But (3) shows that the key condition (b):

\[ \frac{\partial^2 F}{\partial x^2}(0, 0) > 0, \]

is violated, and hence we must make some alternative hypotheses. The hypotheses that we will make will involve the second and the third partial derivatives of \( F \), and also that \( \lambda(\mu) \) really passes through \(-1\), i.e. \( \frac{d\lambda}{d\mu}(0) \neq 0 \).
\[(F_{\mu}^{\circ 2})''(x) = F_{\mu}''(F_{\mu}(x))F_{\mu}'(x)^2 + F_{\mu}'(F_{\mu}(x))F_{\mu}''(x). \quad (2)\]

To understand the hypothesis we will make involving the partial derivatives of \(F\), let us differentiate (2) once more with respect to \(x\) to obtain

\[(F_{\mu}^{\circ 2})'''(x) =
F_{\mu}'''(F_{\mu}(x))F_{\mu}'(x)^3 + 2F_{\mu}''(F_{\mu}(x))F_{\mu}''(x)F_{\mu}'(x)
+ F_{\mu}''(F_{\mu}(x))F_{\mu}'(x)F_{\mu}'''(x) + F_{\mu}'(F_{\mu}(x))F_{\mu}'''(x).
\]

At \((x, \mu) = (0, 0)\) this simplifies to

\[- \left[ 2 \frac{\partial^3 F}{\partial x^3}(0, 0) + 3 \left( \frac{\partial^2 F}{\partial x^2}(0, 0) \right)^2 \right]. \quad (4)\]
**Theorem**

[Period doubling bifurcation.] *Suppose that* $F$ *is* $C^3$, *that*

\[(d) F'_0(0) = -1 \quad (e) \quad \frac{d\lambda}{d\mu}(0) > 0, \quad \text{and} \]

\[(f) \quad 2 \frac{\partial^3 F}{\partial x^3}(0,0) + 3 \left( \frac{\partial^2 F}{\partial x^2}(0,0) \right)^2 > 0. \]

*Then there are non-empty intervals* $(\mu_1, 0)$ *and* $(0, \mu_2)$ *and* $\epsilon > 0$ *so that*

(i) *If* $\mu \in (\mu_1, 0)$ *then* $F_\mu$ *has one repelling fixed point and one attracting orbit of period two in* $(-\epsilon, \epsilon)$

(ii) *If* $\mu \in (0, \mu_2)$ *then* $F^{\circ 2}_\mu$ *has a single fixed point in* $(-\epsilon, \epsilon)$ *which is in fact an attracting fixed point of* $F_\mu$. 
Proof of the period doubling bifurcation theorem, I.

Let

\[ H(x, \mu) := F^\circ 2(x, \mu) - x. \]

Then by the remarks before the statement of the theorem, \( H \) vanishes at the origin together with its first two partial derivatives with respect to \( x \). The expression

\[- \left[ 2 \frac{\partial^3 F}{\partial x^3} (0, 0) + 3 \left( \frac{\partial^2 F}{\partial x^2} (0, 0) \right)^2 \right], \quad (4)\]

(which used condition (d)) together with condition (f) gives

\[ \frac{\partial^3 H}{\partial x^3} (0, 0) < 0. \]
One of the zeros of \( H(x, \mu) := F^2(x, \mu) - x \) at the origin corresponds to the fact that \((0, 0)\) is a fixed point. Let us factor this out: Define \( P(x, \mu) \) by

\[
H(x, \mu) = (x - x(\mu))P(x, \mu).
\] (5)
Proof of the period doubling bifurcation theorem, III.

Then

\[ \frac{\partial H}{\partial x} = P + (x - x(\mu)) \frac{\partial P}{\partial x} \]
\[ \frac{\partial^2 H}{\partial x^2} = 2 \frac{\partial P}{\partial x} + (x - x(\mu)) \frac{\partial^2 P}{\partial x^2} \]
\[ \frac{\partial^3 H}{\partial x^3} = 3 \frac{\partial^2 P}{\partial x^2} + (x - x(\mu)) \frac{\partial^3 P}{\partial x^3}. \]

So \( P \) vanishes at the origin together with its first partial derivative with respect to \( x \), while

\[ \frac{\partial^3 H}{\partial x^3}(0, 0) = 3 \frac{\partial^2 P}{\partial x^2}(0, 0) \]

So

\[ \frac{\partial^2 P}{\partial x^2}(0, 0) < 0. \]
We claim that
\[ \frac{\partial P}{\partial \mu} (0, 0) < 0, \tag{7} \]
so that we can apply the implicit function theorem to \( P(x, \mu) = 0 \) to solve for \( \mu \) as a function of \( x \). This will allow us to determine the fixed points of \( F^{\circ 2}_\mu \) which are not fixed points of \( F_\mu \), i.e. the points of period two. To prove (7) we compute \( \frac{\partial H}{\partial x} \) both from its definition \( H(x, \mu) = F^{\circ 2}(x, \mu) - x \) and from (5) to obtain:
Proof of the period doubling bifurcation theorem, V.

\[
\frac{\partial H}{\partial x} = \frac{\partial F}{\partial x}(F(x, \mu), \mu) \frac{\partial F}{\partial x}(x, \mu) - 1
\]

\[
= P(x, \mu) + (x - x(\mu)) \frac{\partial P}{\partial x}(x, \mu).
\]

Recall that \(x(\mu)\) is the fixed point of \(F_\mu\) and that \(\lambda(\mu) = \frac{\partial F}{\partial x}(x(\mu), \mu)\). So substituting \(x = x(\mu)\) into the preceding equation gives

\[
\lambda(\mu)^2 - 1 = P(x, \mu).
\]

Differentiating with respect to \(\mu\) and setting \(\mu = 0\) gives

\[
\frac{\partial P}{\partial \mu}(0, 0) = 2\lambda(0)\lambda'(0) = -2\lambda'(0)
\]

which is \(< 0\) by (e).
Proof of the period doubling bifurcation theorem, VI.

\[
\frac{\partial P}{\partial \mu}(0, 0) < 0. \quad (7)
\]

By the implicit function theorem, (7) implies that there is a $C^2$ function $\nu(x)$ defined near zero as the unique solution of $P(x, \nu(x)) \equiv 0$. Recall that $P$ and its first derivative with respect to $x$ vanish at $(0, 0)$. We now repeat the arguments of the last lecture:
Proof of the period doubling bifurcation theorem, VII.

We have

$$\nu'(x) = -\frac{\partial P/\partial x}{\partial P/\partial \mu}$$

so

$$\nu'(0) = 0$$

and

$$\nu''(0) = -\frac{\partial^2 P/\partial x^2}{\partial P/\partial \mu}(0, 0) < 0$$

since this time both numerator and denominator are negative. So the curve $\nu$ has the same form as in the proof of the fold bifurcation theorem. This establishes the existence of the (strictly) period two points for $\mu < 0$ and their absence for $\mu > 0$. 
Proof of the period doubling bifurcation theorem, VIII.

We now turn to the question of the stability of the fixed points and the period two points. Condition (e):

\[
\frac{d\lambda}{d\mu}(0) > 0,
\]

together with the fact that \(\lambda(0) = -1\) imply that \(\lambda(\mu) < -1\) for \(\mu < 0\) and \(\lambda(\mu) > -1\) for \(\mu > 0\) so the fixed point is repelling to the left and attracting to the right of the origin. As for the period two points, we wish to show that

\[
\frac{\partial F^2}{\partial x}(x, \nu(x)) < 1
\]

for \(x < 0\).
Proof of the period doubling bifurcation theorem, IX.

We need to show that

\[ \frac{\partial F^2}{\partial x}(x, \nu(x)) < 1 \]

for \( x < 0 \).

Now (3):

\[ \frac{\partial^2 F^2}{\partial x^2}(0, 0) = 0 \]

and \( \nu'(0) = 0 \) imply that 0 is a critical point for this function, and the value at this critical point is \( \lambda(0)^2 = 1 \).

To complete the proof we must show that this critical point is a local maximum. So we must compute the second derivative at the origin.
Proof of the period doubling bifurcation theorem, X.

Calling this function $\phi$ we have

\[
\phi(x) := \frac{\partial F^2}{\partial x}(x, \nu(x))
\]

\[
\phi'(x) = \frac{\partial^2 F^2}{\partial x^2}(x, \nu(x)) + \frac{\partial^2 F^2}{\partial x \partial \mu}(x, \nu(x))\nu'(x)
\]

\[
\phi''(x) = \frac{\partial^3 F^2}{\partial x^3}(x, \nu(x)) + 2\frac{\partial^3 F^2}{\partial x^2 \partial \mu}(x, \nu(x))\nu'(x) + \frac{\partial^3 F^2}{\partial x \partial \mu^2}(x, \nu(x))(\nu'(x))^2 + \frac{\partial^2 F^2}{\partial x \partial \mu}(x, \nu(x))\nu''(x).
\]
Proof of the period doubling bifurcation theorem, XI.

\[ \phi''(x) = \frac{\partial^3 F \circ 2}{\partial x^3}(x, \nu(x)) + 2 \frac{\partial^3 F \circ 2}{\partial x^2 \partial \mu}(x, \nu(x)\nu'(x)) \]

\[ + \frac{\partial^3 F \circ 2}{\partial x \partial \mu^2}(x, \nu(x))\nu'(x))^2 + \frac{\partial^2 F \circ 2}{\partial x \partial \mu}(x, \nu(x))\nu''(x). \]

The middle two terms vanish at 0 since \( \nu'(0) = 0 \). The last term becomes

\[ \frac{d\lambda}{d\mu}(0)\nu''(0) < 0 \]

by condition (e) and the fact that \( \nu''(0) < 0 \). We have computed the first term, i.e. the third partial derivative, in (4) using condition (d) and then (f) implies that this expression is negative. This completes the proof of the period doubling bifurcation.
There are obvious variants on the theorem which involve changing signs in hypotheses (e) and or (f). Thus we may have an attractive fixed point merging with two repelling points of period two to produce a repelling fixed point, and/or the direction of the bifurcation may be reversed.