1. Let $X$ be a compact $n$-dimensional differentiable manifold, and $Y \subset X$ a closed submanifold of dimension $m$. Show that the Euler characteristic $\chi(X \setminus Y)$ of the complement of $Y$ in $X$ is given by

$$\chi(X \setminus Y) = \chi(X) + (-1)^{n-m-1}\chi(Y).$$

Does the same result hold if we do not assume that $X$ is compact, but only that the Euler characteristics of $X$ and $Y$ are finite?

2. Prove that the infinite sum

$$\sum_{p \text{ prime}} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \ldots$$

diverges.

3. Let $h(x)$ be a $C^\infty$ function on the real line $\mathbb{R}$. Find a $C^\infty$ function $u(x,y)$ on an open subset of $\mathbb{R}^2$ containing the $x$-axis such that

$$\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = u^2$$

and $u(x,0) = h(x)$.

4. a) Let $K$ be a field, and let $L = K(\alpha)$ be a finite Galois extension of $K$. Assume that the Galois group of $L$ over $K$ is cyclic, generated by an automorphism sending $\alpha$ to $\alpha + 1$. Prove that $K$ has characteristic $p > 0$ and that $\alpha^p - \alpha \in K$.

b) Conversely, prove that if $K$ is of characteristic $p$, then every Galois extension $L/K$ of degree $p$ arises in this way. (Hint: show that there exists $\beta \in L$ with trace 1, and construct $\alpha$ out of the various conjugates of $\beta$.)
5. For small positive $\alpha$, compute

$$\int_0^\infty \frac{x^\alpha \, dx}{x^2 + x + 1}.$$ 

For what values of $\alpha \in \mathbb{R}$ does the integral actually converge?

6. Let $M \in \mathcal{M}_n(\mathbb{C})$ be a complex $n \times n$ matrix such that $M$ is similar to its complex conjugate $\overline{M}$; i.e., there exists $g \in GL_n(\mathbb{C})$ such that $\overline{M} = gMg^{-1}$. Prove that $M$ is similar to a real matrix $M_0 \in \mathcal{M}_n(\mathbb{R})$. 
QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Wednesday, March 13 (Day 2)

1. Prove the Brouwer fixed point theorem: that any continuous map from the closed $n$-disc $D^n \subset \mathbb{R}^n$ to itself has a fixed point.

2. Find a harmonic function $f$ on the right half-plane $\{ z \in \mathbb{C} \mid \text{Re } z > 0 \}$ satisfying
   \[
   \lim_{x \to 0^+} f(x + iy) = \begin{cases} 1 & \text{if } y > 0 \\ -1 & \text{if } y < 0 \end{cases}.
   \]

3. Let $n$ be any integer. Show that any odd prime $p$ dividing $n^2 + 1$ is congruent to 1 (mod 4).

4. Let $V$ be a vector space of dimension $n$ over a finite field with $q$ elements.
   a) Find the number of one-dimensional subspaces of $V$.
   b) For any $k : 1 \leq k \leq n - 1$, find the number of $k$-dimensional subspaces of $V$.

5. Let $K$ be a field of characteristic 0. Let $\mathbb{P}^N$ be the projective space of homogeneous polynomials $F(X,Y,Z)$ of degree $d$ modulo scalars ($N = d(d+3)/2$). Let $W \subset \mathbb{P}^N$ be the subset of polynomials $F$ of the form
   \[
   F(X,Y,Z) = \prod_{i=1}^{d} L_i(X,Y,Z)
   \]
   for some collection of linear forms $L_1, \ldots, L_d$.
   a. Show that $W$ is a closed subvariety of $\mathbb{P}^N$.
   b. What is the dimension of $W$?
   c. Find the degree of $W$ in case $d = 2$ and in case $d = 3$. 
6. a. Suppose that $M \to \mathbb{R}^{n+1}$ is an embedding of an $n$-dimensional Riemannian manifold (i.e., $M$ is a hypersurface). Define the *second fundamental form* of $M$.

b. Show that if $M \subset \mathbb{R}^{n+1}$ is a compact hypersurface, its second fundamental form is positive definite (or negative definite, depending on your choice of normal vector) at at least one point of $M$. 
1. In $\mathbb{R}^3$, let $S$, $L$ and $M$ be the circle and lines

\[
S = \{(x, y, z) : x^2 + y^2 = 1; \ z = 0\} \\
L = \{(x, y, z) : x = y = 0\} \\
M = \{(x, y, z) : x = \frac{1}{2}; \ y = 0\} \\
\]

respectively.

a. Compute the homology groups of the complement $\mathbb{R}^3 \setminus (S \cup L)$.

b. Compute the homology groups of the complement $\mathbb{R}^3 \setminus (S \cup L \cup M)$.

2. Let $L, M, N \subset \mathbb{P}^3_\mathbb{C}$ be any three pairwise disjoint lines in complex projective threespace. Show that there is a unique quadric surface $Q \subset \mathbb{P}^3_\mathbb{C}$ containing all three.

3. Let $G$ be a compact Lie group, and let $\rho : G \to GL(V)$ be a representation of $G$ on a finite-dimensional $\mathbb{R}$-vector space $V$.

a) Define the dual representation $\rho^* : G \to GL(V^*)$ of $V$.

b) Show that the two representations $V$ and $V^*$ of $G$ are isomorphic.

c) Consider the action of $SO(n)$ on the unit sphere $S^{n-1} \subset \mathbb{R}^n$, and the corresponding representation of $SO(n)$ on the vector space $V$ of $C^\infty$ $\mathbb{R}$-valued functions on $S^{n-1}$. Show that each nonzero irreducible $SO(n)$-subrepresentation $W \subset V$ of $V$ has a nonzero vector fixed by $SO(n-1)$, where we view $SO(n-1)$ as the subgroup of $SO(n)$ fixing the vector $(0, \ldots, 0, 1)$.

4. Show that if $K$ is a finite extension field of $\mathbb{Q}$, and $A$ is the integral closure of $\mathbb{Z}$ in $K$, then $A$ is a free $\mathbb{Z}$-module of rank $[K : \mathbb{Q}]$ (the degree of the field extension). (Hint: sandwich $A$ between two free $\mathbb{Z}$-modules of the same rank.)
5. Let \( n \) be a nonnegative integer. Show that
\[
\sum_{0 \leq k \leq l \atop k+l=n} (-1)^l \binom{l}{k} = \begin{cases} 
1 & \text{if } n \equiv 0 \pmod{3} \\
-1 & \text{if } n \equiv 1 \pmod{3} \\
0 & \text{if } n \equiv 2 \pmod{3}
\end{cases}.
\]
(Hint: Use a generating function.)

6. Suppose \( K \) is integrable on \( \mathbb{R}^n \) and for \( \epsilon > 0 \) define
\[
K_\epsilon(x) = \epsilon^{-n} K\left(\frac{x}{\epsilon}\right).
\]
Suppose that \( \int_{\mathbb{R}^n} K = 1 \).
   a. Show that \( \int_{\mathbb{R}^n} K_\epsilon = 1 \) and that \( \int_{|x|>\delta} |K_\epsilon| \to 0 \) as \( \epsilon \to 0 \).
   b. Suppose \( f \in L^p(\mathbb{R}^n) \) and for \( \epsilon > 0 \) let \( f_\epsilon \in L^p(\mathbb{R}^n) \) be the convolution
\[
f_\epsilon(x) = \int_{y \in \mathbb{R}^n} f(y) K_\epsilon(x-y) dy.
\]
Show that for \( 1 \leq p < \infty \) we have
\[
\|f_\epsilon - f\|_p \to 0 \text{ as } \epsilon \to 0.
\]
   c. Conclude that for \( 1 \leq p < \infty \) the space of smooth compactly supported functions on \( \mathbb{R}^n \) is dense in \( L^p(\mathbb{R}^n) \).
Extra problems: Let me know if you think these should replace any of the ones above, either for balance or just by preference.

1. Suppose that $M \rightarrow \mathbb{R}^N$ is an embedding of an $n$-dimensional manifold into $N$-dimensional Euclidean space. Endow $M$ with the induced Riemannian metric. Let $\gamma : (-1, 1) \rightarrow M$ be a curve in $M$ and $\overline{\gamma} : (-1, 1) \rightarrow \mathbb{R}^N$ be given by composition with the embedding. Assume that $\| \frac{d\gamma}{dt} \| \equiv 1$. Prove that $\gamma$ is a geodesic iff

$$\frac{d^2\overline{\gamma}}{dt^2}$$

is normal to $M$ at $\gamma(t)$ for all $t$.

2. Let $A$ be a commutative Noetherian ring. Prove the following statements and explain their geometric meaning (even if you do not prove all the statements below, you may use any statement in proving a subsequent one):

   a) $A$ has only finitely many minimal prime ideals $\{p_k | k = 1, \ldots, n\}$, and every prime ideal of $A$ contains one of the $p_k$.

   b) $\bigcap_{k=1}^{n} p_k$ is the set of nilpotent elements of $A$.

   c) If $A$ is reduced (i.e., its only nilpotent element is 0), then $\bigcup_{k=1}^{n} p_k$ is the set of zero-divisors of $A$.

4. Let $A$ be the $n \times n$ matrix

$$
\begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
1/n & 1/n & 1/n & \ldots & 1/n
\end{pmatrix}
$$

Prove that as $k \rightarrow \infty$, $A^k$ tends to a projection operator $P$ onto a one-dimensional subspace. Find the kernel and image of $P$. 