Problem 1 (DG)
Let $S$ denote the surface in $\mathbb{R}^3$ where the coordinates $(x, y, z)$ obey $x^2 + y^2 = 1 + z^2$. This surface can be parametrized by coordinates $t \in \mathbb{R}$ and $\theta \in \mathbb{R}/(2\pi \mathbb{Z})$ by the map

$$(t, \theta) \rightarrow \psi(t, \theta) = (\sqrt{1+t^2} \cos \theta, \sqrt{1+t^2} \sin \theta, t).$$

a) Compute the induced inner product on the tangent space to $S$ using these coordinates.

b) Compute the Gaussian curvature of the metric that you computed in Part a).

c) Compute the parallel transport around the circle in $S$ where $z = 0$ for the Levi-Civita connection of the metric that you computed in Part a).

Problem 2 (T)
Let $X$ be path-connected and locally path-connected, and let $Y$ be a finite Cartesian product of circles. Show that if $\pi_1(X)$ is finite, then every continuous map from $X$ to $Y$ is null-homotopic. (Hint: recall that there is a fiber bundle $Z \rightarrow \mathbb{R} \rightarrow S^1$.)

Problem 3 (AN)
Let $K$ be the field $\mathbb{C}(z)$ of rational functions in an indeterminate $z$, and let $F \subset K$ be the subfield $\mathbb{C}(u)$ where $u = (z^6 + 1)/z^3$.

a) Show that the field extension $K/F$ is normal, and determine its Galois group.

b) Find all fields $E$, other than $F$ and $K$ themselves, such that $F \subset E \subset K$. For each $E$, determine whether the extensions $E/F$ and $K/E$ are normal.
PROBLEM 4 (AG)
The nodal cubic is the curve in $\mathbb{CP}^2$ (denoted by $X$) given in homogeneous coordinates $(x, y, z)$ by the locus $\{zy^2 = x^2(x + z)\}$.

a) Give a definition of a rational map between algebraic varieties.
b) Show that there is a birational map from $X$ to $\mathbb{CP}^1$.
c) Explain how to resolve the singularity of $X$ by blowing up a point in $\mathbb{CP}^2$.

PROBLEM 5 (RA)
Let $\mathcal{B}$ and $\mathcal{L}$ denote Banach spaces, and let $\|\cdot\|_{\mathcal{B}}$ and $\|\cdot\|_{\mathcal{L}}$ denote their norms.

a) Let $L: \mathcal{B} \to \mathcal{L}$ denote a continuous, invertible linear map and let $m: \mathcal{B} \otimes \mathcal{B} \to \mathcal{L}$ denote a linear map such that $\|m(\phi \otimes \psi)\|_{\mathcal{L}} \leq \|\phi\|_{\mathcal{B}} \|\psi\|_{\mathcal{B}}$ for all $\phi, \psi \in \mathcal{B}$. Prove the following assertions:
   
   • There exists a number $\kappa > 1$ depending only on $L$ such that if $a \in \mathcal{B}$ has norm less than $\kappa^2$, then there is a unique solution to the equation $L\phi + m(\phi \otimes \phi) = a$ with $\|\phi\|_{\mathcal{B}} < \kappa^{-1}$.
   
   • The norm of the solution from the previous bullet is at most $\kappa \|a\|_{\mathcal{L}}$.

b) Recall that a Banach space is defined to be a complete, normed vector space. Is the assertion of Part a) of the first bullet always true if $\mathcal{B}$ is normed but not complete? If not, explain where the assumption that $\mathcal{B}$ is complete enters your proof of Part a).

PROBLEM 6 (CA)
Fix $a \in \mathbb{C}$ and an integer $n \geq 2$. Show that the equation $az^n + z + 1 = 0$ for a complex number $z$ necessarily has a solution with $|z| \leq 2$. 
PROBLEM 1 (DG)
Let $k$ denote a positive integer. A non-optimal version of the Whitney embedding theorem states that any $k$-dimensional manifold can be embedded into $\mathbb{R}^{2k+1}$. Using this, show that any $k$-dimensional manifold can be immersed in $\mathbb{R}^{2k}$. (Hint: Compose the embedding with a projection onto an appropriate subspace.)

PROBLEM 2 (T)
Let $X$ be a CW-complex with a single cell in each of the dimensions $0, 1, 2, 3, $ and $5$ and no other cells.

a) What are the possible values of $H_*(X; \mathbb{Z})$? (Note: it is not sufficient to consider $H_n(X; \mathbb{Z})$ for each $n$ independently. The value of $H_i(X; \mathbb{Z})$ may constrain the value of $H_2(X; \mathbb{Z})$, for instance.)

b) Now suppose in addition that $X$ is its own universal cover. What extra information does this provide about $H_*(X; \mathbb{Z})$?

PROBLEM 3 (AN)
Let $k$ be a finite field of characteristic $p$, and $n$ a positive integer. Let $G$ be the group of invertible linear transformations of the $k$-vector space $k^n$. Identify $G$ with the group of invertible $n \times n$ matrices with entries in $k$ (acting from the left on column vectors).

a) Prove that the order of $G$ is $\prod_{m=0}^{n-1} (q^n - q^m)$ where $q$ is the number of elements of $k$.

b) Let $U$ be the subgroup of $G$ consisting of upper-triangular matrices with all diagonal entries equal $1$. Prove that $U$ is a $p$-Sylow subgroup of $G$.

c) Suppose $H \subseteq G$ is a subgroup whose order is a power of $p$. Prove that there is a basis $(v_1, v_2, ..., v_n)$ of $k^n$ such that for every $h \in H$ and every $m \in \{1, 2, 3, ..., n\}$, the vector $h(v_m) - v_m$ is in the span of $\{v_d : d < m\}$. 
**Problem 4 (AG)**

Let $X$ be a complete intersection of surfaces of degrees $a$ and $b$ in $\mathbb{CP}^3$. Compute the Hilbert polynomial of $X$.

**Problem 5 (RA)**

Let $C^0$ denote the vector space of continuous functions on the interval $[0, 1]$. Define a norm on $C^0$ as follows: If $f \in C^0$, then its norm (denoted by $\|f\|$) is

$$
\|f\| = \sup_{t \in [0, 1]} |f(t)|.
$$

Let $C^\infty$ denote the space of smooth functions on $[0, 1]$. View $C^\infty$ as a normed, linear space with the norm defined as follows: If $f \in C^\infty$, then its norm (denoted by $\|f\|_*$) is

$$
\|f\|_* = \int_{[0,1]} (|\frac{d}{dt} f(t)| + |f(t)|) \, dt.
$$

a) Prove that $C^0$ is a Banach space with respect to the norm $\|\cdot\|$. In particular, prove that it is complete.

b) Let $\psi$ denote the ‘forgetful’ map from $C^\infty$ to $C^0$ that sends $f$ to $f$. Prove that $\psi$ is a bounded map from $C^\infty$ to $C^0$, but not a compact map from $C^\infty$ to $C^0$.

**Problem 6 (CA)**

Let $\mathbb{D}$ denote the closed disk in $\mathbb{C}$ where $|z| \leq 1$. Fix $R > 0$ and let $\varphi: \mathbb{D} \to \mathbb{C}$ denote a continuous map with the following properties:

i) $\varphi$ is holomorphic on the interior of $\mathbb{D}$.

ii) $\varphi(0) = 0$ and its $z$-derivative, $\varphi'$, obeys $\varphi'(0) = 1$.

iii) $|\varphi| \leq R$ for all $z \in \mathbb{D}$.

Since $\varphi'(0) = 1$, there exists $\delta > 0$ such that $\varphi$ maps the $|z| < \delta$ disk diffeomorphically onto its image. Prove the following:

a) There is a unique solution in $[0, 1]$ to the equation $2R\delta = (1 - \delta)^3$.

b) Let $\delta_*$ denote the unique solution to this equation. If $0 < \delta < \delta_*$, then $\varphi$ maps the $|z| < \delta$ disk diffeomorphically onto its image.
PROBLEM 1 (DG)
Recall that a symplectic manifold is a pair $(M, \omega)$, where $M$ is a smooth manifold and $\omega$ is a closed nondegenerate differential 2-form on $M$. (The 2-form $\omega$ is called the symplectic form.)

a) Show that if $H: M \to \mathbb{R}$ is a smooth function, then there exists a unique vector field, to be denoted by $X_H$, satisfying $\iota_{X_H} \omega = dH$. (Here, $\iota$ denotes the contraction operation.)

b) Supposing that $t > 0$ is given, suppose in what follows that the flow of $X_H$ is defined for time $t$, and let $\phi_t$ denote the resulting diffeomorphism of $M$. Show that $\phi_t^* \omega = \omega$.

c) Denote the Euclidean coordinates on $\mathbb{R}^4$ by $(x_1, y_1, x_2, y_2)$ and use these to define the symplectic form $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Find a function $H: \mathbb{R}^4 \to \mathbb{R}$ such that the diffeomorphism $\phi_{t=1}$ that is defined by the time $t = 1$ flow of $X_H$ fixes the half space where $x_1 \leq 0$ and moves each point in the half space where $x_1 \geq 1$ by 1 in the $y_2$ direction.

PROBLEM 2 (T)
Let $X$ denote a finite CW complex and let $f: X \to X$ be a self-map of $X$. Recall that the Lefschetz trace of $f$, denoted by $\tau(f)$, is defined by the rule

$$\tau(f) = \sum_{n=0}^{\infty} (-1)^n \text{tr}(f_n: H_n(X; \mathbb{Q}) \to H_n(X; \mathbb{Q}))$$

with $f_n$ denoting the induced homomorphism. Use $\tau(\cdot)$ to answer the following:

a) Does there exist a continuous map from $\mathbb{RP}^2$ to itself with no fixed points? If so, give an example; and if not, give a proof.

b) Does there exist a continuous map from $\mathbb{RP}^3$ to itself with no fixed points? If so, give an example; and if not, give a proof.

PROBLEM 3 (AN)
Let $A$ be the ring $\mathbb{Z}[\sqrt[5]{2016}] = \mathbb{Z}[X]/(X^5 - 2016)$. Given that 2017 is prime in $\mathbb{Z}$, determine the factorization of $2017 \cdot A$ into prime ideals of $A$. 
PROBLEM 4 (AG)
a) State a version of the Riemann–Roch theorem.
b) Apply this theorem to show that if X is a complete nonsingular curve and P ∈ X is any point, there is a rational function on X which has a pole at P and is regular on X−{P}.

PROBLEM 5 (RA)
Let ℘ denote a probability measure for a real valued random variable with mean 0. Denote this random variable by x. Suppose that the random variable |x| has mean equal to 2.
a) Given R > 2, state a non-trivial upper bound for event that x ≥ R. (The trivial upper bound is 1.)
b) Give a non-zero lower bound for the standard deviation of x.
c) A function f on ℝ is Lipshitz when there exists a number c ≥ 0 such that

[f(p) - f(p')] ≤ c |p - p'| for any pair p, p' ∈ ℝ.

Let ˆφ denote the function on ℝ whose value at a given p ∈ ℝ is the expectation of the random variable e^{ipx}. (This is the characteristic function of ℘.) Give a rigorous proof that ˆφ is Lipshitz and give an upper bound for c in this case.
d) Suppose that the standard deviation of x is equal to 4. Let N denote an integer greater than 1, and let \{x₁, …, x_N\} denote a set of independent random variables each with probabilities given by ℘. Use S_N to denote the random variable \(\frac{1}{N} (x₁ + ⋯ + x_N)\).
The central limit theorem gives an integral that approximates the probability of the event where \(S_N \in [-1,1]\) when N is large. Write this integral.

PROBLEM 6 (CA)
Let H ⊂ ℂ denote the open right half plane, thus H = \{z = x + iy: x > 0\}. Suppose that f: H → ℂ is a bounded, analytic function such that f(1/n) = 0 for each positive integer n. Prove that f(z) = 0 for all z.
(Hint: Consider the behavior of the sequence of functions \{h_N(z) = \prod_{n=1}^{N} \frac{z-1/n}{z+1/n}\}_{n=1,2…} on H and, in particular, on the positive real axis.)