1. (AG) Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree $d$.

(a) Let $K_C$ be the canonical bundle of $C$. For what integer $n$ is it the case that $K_C \cong \mathcal{O}_C(n)$?

(b) Prove that if $d \geq 4$ then $C$ is not hyperelliptic.

(c) Prove that if $d \geq 5$ then $C$ is not trigonal (that is, expressible as a 3-sheeted cover of $\mathbb{P}^1$).

Solution: By the adjunction formula, the canonical divisor class is $K_C = \mathcal{O}_C(d-3)$, that is, plane curves of degree $d-3$ cut out canonical divisors on $C$. It follows that if $d \geq 4$ then any two points $p, q \in C$ impose independent conditions on the canonical series $|K_C|$; that is, $h^0(K_C(-p- q)) = g - 2$, so by Riemann-Roch $h^0(\mathcal{O}_C(p+q)) = 1$, i.e., $C$ is not hyperelliptic. Similarly, if $d \geq 5$ then any three points $p, q, r \in C$ impose independent conditions on the canonical series $|K_C|$; by Riemann-Roch it follows that $h^0(\mathcal{O}_C(p+q+r)) = 1$, so $C$ is not trigonal.

2. (A) Let $S_4$ be the group of automorphisms of a 4-element set. Give the character table for $S_4$ and explain how you arrived at it.

Solution: To start with, there are five conjugacy classes in $S_4$: (1), (12), (123), (1234) and (12)(34). The characters of the trivial and alternating representations $U$ and $U'$ are clear. The standard representation of $S_4$ on $\mathbb{C}^4$ splits as a direct sum of the trivial and a three-dimensional representation $V$, whose character is simply the character of $\mathbb{C}^4$ minus one; we see that it’s irreducible because the norm of its character is 1. We get another irreducible as $V' = V \otimes U'$; its character is $\chi_{V'} = \chi_V \chi_U$. The final irreducible representation $W$ (and its character) can be found by pulling back the standard representation of $S_3$ via the quotient map $S_4 \to S_3$ (or by the orthogonality relations). Altogether, we have
3. (DG) Let 
\[ M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 - y^2 - z^3 - z = 0\}. \]

(a) Prove that \( M \) is a smooth surface in \( \mathbb{R}^3 \).

(b) For what values of \( c \in \mathbb{R} \) does the plane \( z = c \) intersect \( M \) transversely?

Solution: See attached.

4. Define the Banach space \( \mathcal{L} \) to be the completion of the space of continuous functions on the interval \([-1, 1] \subset \mathbb{R}\) using the norm
\[ \|f\| = \int_{-1}^{1} |f(t)| dt. \]

Suppose that \( f \in \mathcal{L} \) and \( t \in [-1, 1] \). For \( h > 0 \), let \( I_h \) be the set of points in \([-1, 1]\) with distance \( h \) or less from \( t \). Prove that
\[ \lim_{h \to 0} \int_{t \in I_h} |f(t)| dt = 0 \]

Solution: See attached.

5. (AT) What are the homology groups of the 5-manifold \( \mathbb{R}P^2 \times \mathbb{R}P^3 \),

(a) with coefficients in \( \mathbb{Z} \)?

(b) with coefficients in \( \mathbb{Z}/2 \)?

(c) with coefficients in \( \mathbb{Z}/3 \)?

Solution: \( \mathbb{R}P^2 \) and \( \mathbb{R}P^3 \) have cell complexes with sequences
\[ 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0 \quad \text{and} \quad 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0 \]
where the maps are alternately 0 and multiplication by 2; from this the homology groups of $\mathbb{R}P^2$ and $\mathbb{R}P^3$ can be calculated as $\mathbb{Z}, \mathbb{Z}/2, 0$ and $\mathbb{Z}, \mathbb{Z}/2, 0, \mathbb{Z}$ respectively. The rest is just Kunneth; the answers are

(a): $\mathbb{Z}, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^2, \mathbb{Z}, \mathbb{Z}/2, 0$;
(b): $\mathbb{Z}/2, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^3, (\mathbb{Z}/2)^3, (\mathbb{Z}/2)^2, \mathbb{Z}/2$;
(c): $\mathbb{Z}/3, 0, 0, \mathbb{Z}/3, 0, 0$

6. Let $\Omega$ be an open subset of the Euclidean plane $\mathbb{R}^2$. A map $f: \Omega \to \mathbb{R}^2$ is said to be conformal at $p \in \Omega$ if its differential $df_p$ preserves the angle between any two tangent vectors at $p$. Now view $\mathbb{R}^2$ as $\mathbb{C}$ and a map $f: \Omega \to \mathbb{R}^2$ as a $\mathbb{C}$-valued function on $\Omega$.

(a) Supposing that $f$ is a holomorphic function on $\Omega$, prove that $f$ is conformal where its differential is nonzero.

(b) Suppose that $f$ is a nonconstant holomorphic function on $\Omega$, and $p \in \Omega$ is a point where $df_p = 0$. Let $L_1$ and $L_2$ denote distinct lines through $p$. Prove that the angle at $f(p)$ between $f(L_1)$ and $f(L_2)$ is $n$ times that between $L_1$ and $L_2$, with $n$ being an integer greater than 1.

Solution: See attached.
1. (AT) Let $X \subset \mathbb{R}^3$ be the union of the unit sphere $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ and the line segment $I = \{(x, 0, 0) \mid -1 \leq x \leq 1\}$.

(a) What are the homology groups of $X$?
(b) What are the homotopy groups $\pi_1(X)$ and $\pi_2(X)$?

Solution: Under the attaching map $I \hookrightarrow X$, the boundary $\phi(I)$ is homologous to 0, so attaching $I$ simply adds one new, non-torsion generator to $H^1$; thus $H_0(X) = H^1(X) = H^2(X) = \mathbb{Z}$, and all other homology groups are 0. Similarly, $\pi_1(X) = \mathbb{Z}$. For $\pi_2(X)$, note that the universal cover of $X$ is a string of spheres attached in a sequence by line segments; $\pi_2(X)$ is thus the free abelian group on countably many generators.

2. (A) Let $f(t) = t^4 + bt^2 + c \in \mathbb{Z}[t]$.

(a) If $E$ is the splitting field for $f$ over $\mathbb{Q}$, show that $Gal(E/\mathbb{Q})$ is isomorphic to a subgroup of the dihedral group $D_8$.
(b) Given an example of $b$ and $c$ for which $f$ is irreducible, and for which the Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Justify.
(c) Give an example of $b$ and $c$ for which $f$ is irreducible, and for which the Galois group is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. Justify.
(d) Give an example of $b$ and $c$ for which $f$ is irreducible, and for which the Galois group is isomorphic to $D_8$.

Solution:

(a) Obviously if $\alpha$ is a root of $f$, so is $-\alpha$. So let $\pm \alpha_1, \pm \alpha_2$ be the four distinct roots of $f$ in $E$. If $\phi$ is an element of the Galois group, it must permute the roots of $f$—moreover, $\phi$ is determined completely by its action on $\alpha_1$ and $\alpha_2$. Also by definition of automorphism, note that $\phi(\alpha_1)$ cannot be a rational multiple of $\phi(\alpha_2)$, while $\phi(-\alpha_1) = -\phi(\alpha_1)$. Hence any field automorphism must necessarily give rise to a symmetry of the following square:

$$
\begin{array}{cc}
\alpha_1 & \alpha_2 \\
-\alpha_2 & -\alpha_1 \\
\end{array}
$$
This gives the injection of $\text{Gal}$ into $D_8$.

(b) An obvious strategy is to find a quadratic extension of a quadratic extension, then find an element whose minimal polynomial is degree 4. For instance, the element $\alpha = \sqrt{2} + \sqrt{3}$ in $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ has a degree 4 minimal polynomial which we can construct by repeatedly multiplying by conjugates: Begin with $t - \sqrt{2} - \sqrt{3}$, the multiply by $(t - \sqrt{2}) + \sqrt{3}$, then multiply this by $(t^2 + \sqrt{2})^2 - 3$. For this choice of $\alpha$, we have $f(t) = t^4 - 10t^2 + 1$.

(c) Taking $b = 0$ and $c = 1$, we see that the splitting field is isomorphic to the subfield of the complex numbers generated by adjoining to $\mathbb{Q}$ the number $\alpha = e^{\pi i/4}$. This is a degree 4 field over $\mathbb{Q}$. Since we have a splitting field in characteristic zero, the Galois group has order 4. We see that the field automorphism sending

$$\alpha \mapsto \alpha^3$$

has order 4, hence the Galois group is cyclic.

(d) Take $b = 0$ and $c = 2$. Clearly we have roots $\alpha_1 = 2^{1/4}$ and $\alpha_2 = i2^{1/4}$, which together lie in an extension of at least degree 8 over $\mathbb{Q}$. By part (a), the Galois group must be $D_8$ itself.

3. (CA) Let $a \in (0, 1)$. By using a contour integral, compute

$$\int_{0}^{2\pi} \frac{dx}{1 - 2a \cos x + a^2}.$$ 

Solution (HT): By the periodicity of $\cos$, it suffices to compute the integral from $-\pi$ to $\pi$. We note that there is a pole for the function

$$f(z) = \frac{1}{1 - 2a \cos z + a^2}$$

at $z_0 = i \cosh^{-1} \frac{1 + a^2}{2a}$. Let $R_t$ be the rectangle bordered by the lines $x = \pm \pi$ and $y = 0, y = t$. As $t \to \infty$, the contribution from the line $y = t$ goes to zero. On the other hand, for all values of $t$, the contribution to the integral from $x = \pm \pi$ cancel each other out. Thus the integral along the bottom edge of the rectangle (which is what we seek) is equal to $2\pi i$ times the residue of $f(z)$ at $z_0$. Near $z_0$, we have that

$$1 - 2a \cos z + a^2 = (z - z_0)2ai \sinh iz_0 + \ldots$$

so we conclude the integral is given by

$$\frac{2\pi i}{2ai \sinh z_0}.$$ 

This simplifies to

$$\frac{2\pi}{1 - a^2}.$$
Alternate solution (CH): Write the integral as a contour integral on the unit circle: set $dx = -idz/z$, so that

$$
\int_0^{2\pi} \frac{1}{1 - 2a \cos x + a^2} dx = -i \int_{|z|=1} \frac{1}{z(1 + a^2) - az^2 - a} dz.
$$

Factor the denominator to find the poles of the latter integrand; one is inside the unit circle and one outside. Calculate the residue at the former pole and use Cauchy’s theorem to evaluate the integral.

4. (AG) Let $K$ be an algebraically closed field of characteristic 0 and let $Q \subset \mathbb{P}^n$ be a smooth quadric hypersurface over $K$.

(a) Show that $Q$ is rational by exhibiting a birational map $\pi : Q \to \mathbb{P}^{n-1}$.

(b) How does the map $\pi$ factor into blow-ups and blow-downs?

Solution: For the first part, we choose any point $p \in Q$ and take $\pi$ to be the projection from $p$. Since $Q$ has degree 2, a general line in $\mathbb{P}^n$ through $p$ will meet $Q$ in one other point, so that the map $\pi : Q \to \mathbb{P}^{n-1}$ has degree 1; that is, it is a birational map. This map blows up the point $p$, and then blows down the union of the lines on $Q$ through $p$. In the other direction, starting with $\mathbb{P}^{n-1}$ we blow up the intersection $Z = S \cap H$ of a quadric hypersurface $S \subset \mathbb{P}^{n-1}$ and a hyperplane $H \subset \mathbb{P}^{n-1}$, and then blow down the proper transform of $H$.

5. DG Let $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ be the unit sphere centered at the origin in $\mathbb{R}^3$.

(a) Prove that the vector field

$$
v = yz \frac{\partial}{\partial x} + zx \frac{\partial}{\partial y} - 2xy \frac{\partial}{\partial z}
$$

on $\mathbb{R}^3$ is tangent to $S$ at all points of $S$, and thus defines a section of the tangent bundle $TS$.

(b) Let $g$ be the metric on $S$ induced from the euclidean metric on $\mathbb{R}^3$, and let $\nabla$ be the associated, metric compatible, torsion free covariant derivative. The tensor $\nabla v$ is a section of $TS \otimes TS^*$. Write $\nabla v$ at the point $(0,0,1) \in S$ using the coordinates $(x_1,x_2)$ given by the map $(x_1,x_2) \mapsto (x_1,x_2,\sqrt{1-x_1^2-x_2^2})$ from the unit disc $x_1^2 + x_2^2 < 1$ to $S$.

Solution: See attached

6. (RA) Let $L$ be a positive real number.
(a) Compute the Fourier expansion of the function $x$ on the interval $[-L, L] \subset \mathbb{R}$.

(b) Prove that the Fourier transform does not converge to $x$ pointwise on the closed interval $[-L, L]$.

Solution: See attached. One note: the second part follows immediately from the observation that whatever the Fourier expansion converges to at $-L$ must be the same as what it converges to at $L$. 
1. (DG) The helicoid is the parametrized surface given by

\[ \phi : \mathbb{R}^2 \to \mathbb{R}^3 : (u, v) \to (v \cos u, v \sin u, au) \]

where \( a \) is a real constant. Compute its induced metric.

Solution. Compute \( \frac{\partial}{\partial u} \) and \( \frac{\partial}{\partial v} \) and deduce that the metric is

\[ g = (v^2 + a^2) \, du \otimes du + dv \otimes dv. \]

2. (RA) A real valued function defined on an interval \((a, b) \subset \mathbb{R}\) is said to be convex if

\[ f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) \]

whenever \( x, y \in (a, b) \) and \( t \in (0, 1) \).

(a) Give an example of a non-constant, non-linear convex function.

(b) Prove that if \( f \) is a non-constant convex function on \((a, b) \subset \mathbb{R}\), then the set of local minima of \( f \) is a connected set where \( f \) is constant.

Solution: See attached

3. (AG) Let \( K \) be an algebraically closed field of characteristic 0, and let \( \mathbb{P}^n \) be the projective space of homogeneous polynomials of degree \( n \) in two variables over \( K \). Let \( X \subset \mathbb{P}^n \) be the locus of \( n^{th} \) powers of linear forms, and let \( Y \subset \mathbb{P}^n \) be the locus of polynomials with a multiple root (that is, a repeated factor).

(a) Show that \( X \) and \( Y \subset \mathbb{P}^n \) are closed subvarieties.

(b) What is the degree of \( X \)?

(c) What is the degree of \( Y \)?

Solution: First, \( X \) is the image of the map \( \mathbb{P}^1 \to \mathbb{P}^n \) sending \([a, b] \in \mathbb{P}^1\) to \((ax + by)^n \in \mathbb{P}^n\). This is projectively equivalent (in characteristic 0!) to the degree \( n \) Veronese map, whose image is a closed curve of degree \( n \). Second, \( Y \) is the zero locus of the discriminant, which is a polynomial of degree \( 2n - 2 \) in the coefficients of a polynomial of degree \( n \) (this number can be deduced from the Riemann-Hurwitz formula, which says that a degree \( n \) map from \( \mathbb{P}^1 \) to \( \mathbb{P}^1 \) has \( 2n - 2 \) branch points; that is, a general line in \( \mathbb{P}^n \) meets \( Y \) in \( 2n - 2 \) points).
4. (AT) Let $X$ be a compact, connected and locally simply connected Hausdorff space, and let $p : \tilde{X} \to X$ be its universal covering space. Prove that $\tilde{X}$ is compact if and only if the fundamental group $\pi_1(X)$ is finite.

Solution: See attached

5. (CA) Prove that if $f$ and $g$ are entire holomorphic functions and $|f| \leq |g|$ everywhere, then $f = \alpha \cdot g$ for some complex number $\alpha$.

Solution: The conclusion trivially holds in the case $g = 0$; from now on, assume that $g$ is not the zero function. The identity theorem implies that the zeros of $g$ are isolated, so $h := f/g$ is meromorphic. The function $h$ is bounded by hypothesis, so Riemann’s theorem implies that $h$ can be extended to an entire bounded function. Liouville’s theorem implies that $h$ is constant, which implies the conclusion.

6. (A) Consider the rings

$$R = \mathbb{Z}[x]/(x^2 + 1) \quad \text{and} \quad S = \mathbb{Z}[x]/(x^2 + 5).$$

(a) Show that $R$ is a principal ideal domain.

(b) Show that $S$ is not a principal ideal domain, by exhibiting a non-principal ideal.

Solution: For the first, the fact that $R$ is a principal ideal domain follows from the fact that it’s a Euclidean domain, with size function $|z|^2$: for any $a, b \in R$ we can write

$$b = ma + r$$

with $|r| < |a|$; carrying this out repeatedly shows that the ideal generated by two elements of $R$ can be generated by one. For the second, the ideal $(2, 1 + x) \subset S$ is not principal.