QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Tuesday January 20, 2015 (Day 1)

1. (AG) Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree $d$.
   
   (a) Let $K_C$ be the canonical bundle of $C$. For what integer $n$ is it the case that $K_C \cong \mathcal{O}_C(n)$?
   
   (b) Prove that if $d \geq 4$ then $C$ is not hyperelliptic.
   
   (c) Prove that if $d \geq 5$ then $C$ is not trigonal (that is, expressible as a 3-sheeted cover of $\mathbb{P}^1$).

2. (A) Let $S_4$ be the group of automorphisms of a 4-element set. Give the character table for $S_4$ and explain how you arrived at it.

3. (DG) Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 - y^2 - z^3 - z = 0\}$.
   
   (a) Prove that $M$ is a smooth surface in $\mathbb{R}^3$.
   
   (b) For what values of $c \in \mathbb{R}$ does the plane $z = c$ intersect $M$ transversely?

4. (RA) Define the Banach space $\mathcal{L}$ to be the completion of the space of continuous functions on the interval $[-1, 1] \subset \mathbb{R}$ using the norm

$$ ||f|| = \int_{-1}^{1} |f(t)|dt. $$

Suppose that $f \in \mathcal{L}$ and $t \in [-1, 1]$. For $h > 0$, let $I_h$ be the set of points in $[-1, 1]$ with distance $h$ or less from $t$. Prove that

$$ \lim_{h \to 0} \int_{t \in I_h} |f(t)|dt = 0 $$

5. (AT) What are the homology groups of the 5-manifold $\mathbb{RP}^2 \times \mathbb{RP}^3$,?
   
   (a) with coefficients in $\mathbb{Z}$?
   
   (b) with coefficients in $\mathbb{Z}/2$?
   
   (c) with coefficients in $\mathbb{Z}/3$?

6. (CA) Let $\Omega$ be an open subset of the Euclidean plane $\mathbb{R}^2$. A map $f : \Omega \to \mathbb{R}^2$ is said to be conformal at $p \in \Omega$ if its differential $df_p$ preserves the angle between any two tangent vectors at $p$. Now view $\mathbb{R}^2$ as $\mathbb{C}$ and a map $f : \Omega \to \mathbb{R}^2$ as a $\mathbb{C}$-valued function on $\Omega$. 
(a) Supposing that \( f \) is a holomorphic function on \( \Omega \), prove that \( f \) is conformal where its differential is nonzero.

(b) Suppose that \( f \) is a nonconstant holomorphic function on \( \Omega \), and \( p \in \Omega \) is a point where \( df_p = 0 \). Let \( L_1 \) and \( L_2 \) denote distinct lines through \( p \). Prove that the angle at \( f(p) \) between \( f(L_1) \) and \( f(L_2) \) is \( n \) times that between \( L_1 \) and \( L_2 \), with \( n \) being an integer greater than 1.
1. (AT) Let $X \subset \mathbb{R}^3$ be the union of the unit sphere $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ and the line segment $I = \{(x, 0, 0) \mid -1 \leq x \leq 1\}$.

(a) What are the homology groups of $X$?
(b) What are the homotopy groups $\pi_1(X)$ and $\pi_2(X)$?

2. (A) Let $f(t) = t^4 + bt^2 + c \in \mathbb{Z}[t]$.

(a) If $E$ is the splitting field for $f$ over $\mathbb{Q}$, show that $Gal(E/\mathbb{Q})$ is isomorphic to a subgroup of the dihedral group $D_8$.
(b) Given an example of $b$ and $c$ for which $f$ is irreducible, and for which the Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Justify.
(c) Give an example of $b$ and $c$ for which $f$ is irreducible, and for which the Galois group is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. Justify.
(d) Give an example of $b$ and $c$ for which $f$ is irreducible, and for which the Galois group is isomorphic to $D_8$.

3. (CA) Let $a \in (0, 1)$. By using a contour integral, compute

$$\int_0^{2\pi} \frac{dx}{1 - 2a \cos x + a^2}.$$

4. (AG) Let $K$ be an algebraically closed field of characteristic 0 and let $Q \subset \mathbb{P}^n$ be a smooth quadric hypersurface over $K$.

(a) Show that $Q$ is rational by exhibiting a birational map $\pi : Q \to \mathbb{P}^{n-1}$.
(b) How does the map $\pi$ factor into blow-ups and blow-downs?

5. (DG) Let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

be the unit sphere centered at the origin in $\mathbb{R}^3$.

(a) Prove that the vector field

$$v = yz \frac{\partial}{\partial x} + zx \frac{\partial}{\partial y} - 2xy \frac{\partial}{\partial z}$$

on $\mathbb{R}^3$ is tangent to $S$ at all points of $S$, and thus defines a section of the tangent bundle $TS$. 

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(b) Let \( g \) be the metric on \( S \) induced from the euclidean metric on \( \mathbb{R}^3 \), and let \( \nabla \) be the associated, metric compatible, torsion free covariant derivative. The tensor \( \nabla v \) is a section of \( TS \otimes TS^* \). Write \( \nabla v \) at the point \((0,0,1) \in S\) using the coordinates \((x_1, x_2)\) given by the map \((x_1, x_2) \mapsto (x_1, x_2, \sqrt{1 - x_1^2 - x_2^2})\) from the unit disc \( x_1^2 + x_2^2 < 1 \) to \( S \).

6. (RA) Let \( L \) be a positive real number.

(a) Compute the Fourier expansion of the function \( x \) on the interval \([-L, L] \subset \mathbb{R}\).

(b) Prove that the Fourier transform does not converge to \( x \) pointwise on the closed interval \([-L, L]\).
1. (DG) The helicoid is the parametrized surface given by
   \[ \phi : \mathbb{R}^2 \rightarrow \mathbb{R}^3 : (u, v) \mapsto (v \cos u, v \sin u, au) \]
   where \( a \) is a real constant. Compute its induced metric.

2. (RA) A real valued function defined on an interval \((a, b) \subset \mathbb{R}\) is said to be convex if
   \[ f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) \]
   whenever \( x, y \in (a, b) \) and \( t \in (0, 1) \).
   (a) Give an example of a non-constant, non-linear convex function.
   (b) Prove that if \( f \) is a non-constant convex function on \((a, b) \subset \mathbb{R}\), then the set of local minima of \( f \) is a connected set where \( f \) is constant.

3. (AG) Let \( K \) be an algebraically closed field of characteristic 0, and let \( \mathbb{P}^n \) be the projective space of homogeneous polynomials of degree \( n \) in two variables over \( K \). Let \( X \subset \mathbb{P}^n \) be the locus of \( n^{th} \) powers of linear forms, and let \( Y \subset \mathbb{P}^n \) be the locus of polynomials with a multiple root (that is, a repeated factor).
   (a) Show that \( X \) and \( Y \subset \mathbb{P}^n \) are closed subvarieties.
   (b) What is the degree of \( X \)?
   (c) What is the degree of \( Y \)?

4. (AT) Let \( X \) be a compact, connected and locally simply connected Hausdorff space, and let \( p : \tilde{X} \rightarrow X \) be its universal covering space. Prove that \( \tilde{X} \) is compact if and only if the fundamental group \( \pi_1(X) \) is finite.

5. (CA) Prove that if \( f \) and \( g \) are entire holomorphic functions and \( |f| \leq |g| \) everywhere, then \( f = \alpha \cdot g \) for some complex number \( \alpha \).

6. (A) Consider the rings
   \[ R = \mathbb{Z}[x]/(x^2 + 1) \quad \text{and} \quad S = \mathbb{Z}[x]/(x^2 + 5). \]
   (a) Show that \( R \) is a principal ideal domain.
   (b) Show that \( S \) is not a principal ideal domain, by exhibiting a non-principal ideal.