1. Let \((X, \mu)\) be a measure space with \(\mu(X) < \infty\). For \(q > 0\), let \(L^q = L^q(X, \mu)\) denote the Banach space completion of the space of bounded functions on \(X\) with the norm

\[ ||f||_q = \left( \int_X |f|^q \mu \right)^{\frac{1}{q}}. \]

Now suppose that \(0 < p \leq q\). Prove that all functions in \(L^q\) are in \(L^p\), and that the inclusion map \(L^q \hookrightarrow L^p\) is continuous.

2. Let \(X \subset \mathbb{P}^n\) be an irreducible projective variety of dimension \(k\), \(G(\ell, n)\) the Grassmannian of \(\ell\)-planes in \(\mathbb{P}^n\) for some \(\ell < n - k\), and \(C(X) \subset G(\ell, n)\) the variety of \(\ell\)-planes meeting \(X\). Prove that \(C(X)\) is irreducible, and find its dimension.

3. Let \(\lambda\) be real number greater than 1. Show that the equation \(ze^{\lambda-z} = 1\) has exactly one solution \(z\) with \(|z| < 1\), and that this solution \(z\) is real. (Hint: use Rouché’s theorem.)

4. Let \(k\) be a finite field, with algebraic closure \(\overline{k}\).
   
   (a) For each integer \(n \geq 1\), show that there is a unique subfield \(k_n \subset \overline{k}\) containing \(k\) and having degree \(n\) over \(k\).
   
   (b) Show that \(k_n\) is a Galois extension of \(k\), with cyclic Galois group.
   
   (c) Show that the norm map \(k_n^* \rightarrow k^*\) (sending a nonzero element of \(k_n\) to the product of its Galois conjugates) is a surjective homomorphism.

5. Suppose \(\omega\) is a closed 2-form on a manifold \(M\). For every point \(p \in M\), let

\[ R_p(\omega) = \{ v \in T_pM : \omega_p(v, u) = 0 \text{ for all } u \in T_pM \}. \]

Suppose that the dimension of \(R_p\) is the same for all \(p\). Show that the assignment \(p \mapsto R_p\) as \(p\) varies in \(M\) defines an integrable subbundle of the tangent bundle \(TM\).

6. Let \(X\) be a topological space. We say that two covering spaces \(f : Y \rightarrow X\) and \(g : Z \rightarrow X\) are isomorphic if there exists a homeomorphism \(h : Y \rightarrow Z\) such that \(g \circ h = f\). If \(X\) is a compact oriented surface of genus \(g\) (that is, a \(g\)-holed torus), how many connected 2-sheeted covering spaces does \(X\) have, up to isomorphism?
QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Wednesday January 20, 2010 (Day 2)

1. Let $a$ be an arbitrary real number and $b$ a positive real number. Evaluate the integral
\[ \int_{0}^{\infty} \frac{\cos(ax)}{\cosh(bx)} \, dx \]
(Recall that $\cosh(x) = \cos(ix) = \frac{1}{2}(e^x + e^{-x})$ is the hyperbolic cosine.)

2. For any irreducible plane curve $C \subset \mathbb{P}^2$ of degree $d > 1$, we define the Gauss map $g : C \to \mathbb{P}^2^*$ to be the rational map sending a smooth point $p \in C$ to its tangent line; we define the dual curve $C^* \subset \mathbb{P}^2^*$ of $C$ to be the image of $g$.

(a) Show that the dual of the dual of $C$ is $C$ itself.

(b) Show that two irreducible conic curves $C, C' \subset \mathbb{P}^2$ are tangent if and only if their duals are.

3. Let $\Lambda_1$ and $\Lambda_2 \subset \mathbb{R}^4$ be complementary 2-planes, and let $X = \mathbb{R}^4 \setminus (\Lambda_1 \cup \Lambda_2)$ be the complement of their union. Find the homology and cohomology groups of $X$ with integer coefficients.

4. Let $X = \{(x, y, z) : x^2 + y^2 = 1\} \subset \mathbb{R}^3$ be a cylinder. Show that the geodesics on $X$ are helices, that is, curves such that the angle between their tangent lines and the vertical is constant.

5. (a) Show that if every closed and bounded subspace of a Hilbert space $E$ is compact, then $E$ is finite dimensional.

(b) Show that any decreasing sequence of nonempty, closed, convex, and bounded subsets of a Hilbert space has a nonempty intersection.

(c) Show that any closed, convex, and bounded subset of a Hilbert space is the intersection of the closed balls that contain it.

(d) Deduce that any closed, convex, and bounded subset of a Hilbert space is compact in the weak topology.

6. Let $p$ be a prime, and let $G$ be the group $\mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$.

(a) How many subgroups of order $p$ does $G$ have?

(b) How many subgroups of order $p^2$ does $G$ have? How many of these are cyclic?
1. Consider the ring

\[ A = \mathbb{Z}[x]/(f) \text{ where } f = x^4 - x^3 + x^2 - 2x + 4. \]

Find all prime ideals of \( A \) that contain the ideal \((3)\).

2. Let \( f \) be a holomorphic function on a domain containing the closed disc \( \{ z : |z| \leq 3 \} \), and suppose that

\[ f(1) = f(i) = f(-1) = f(-i) = 0. \]

Show that

\[ |f(0)| \leq \frac{1}{80} \max_{|z|=3} |f(z)| \]

and find all such functions for which equality holds in this inequality.

3. Let \( f: \mathbb{R} \rightarrow \mathbb{R}^+ \) be a differentiable, positive real function. Find the Gaussian curvature and mean curvature of the surface of revolution

\[ S = \{ (x, y, z) : y^2 + z^2 = f(x) \}. \]

4. Show that for any given natural number \( n \), there exists a finite Borel measure on the interval \([0, 1] \subset \mathbb{R}\) such that for any real polynomial of degree at most \( n \), we have

\[ \int_0^1 p \, d\mu = p'(0). \]

Show, on the other hand, that there does not exist a finite Borel measure on the interval \([0, 1] \subset \mathbb{R}\) such that for any real polynomial we have

\[ \int_0^1 p \, d\mu = p'(0). \]

5. Let \( X = \mathbb{RP}^2 \times \mathbb{RP}^4 \).

(a) Find the homology groups \( H_*(X, \mathbb{Z}/2) \)

(b) Find the homology groups \( H_*(X, \mathbb{Z}) \)

(c) Find the cohomology groups \( H^*(X, \mathbb{Z}) \)
6. By a **twisted cubic curve** we mean the image of the map \( \mathbb{P}^1 \to \mathbb{P}^3 \) given by

\[
[X, Y] \mapsto [F_0(X, Y), F_1(X, Y), F_2(X, Y), F_3(X, Y)]
\]

where \( F_0, \ldots, F_3 \) form a basis for the space of homogeneous cubic polynomials in \( X \) and \( Y \).

(a) Show that if \( C \subset \mathbb{P}^3 \) is a twisted cubic curve, then there is a 3-dimensional vector space of homogeneous quadratic polynomials on \( \mathbb{P}^3 \) vanishing on \( C \).

(b) Show that \( C \) is the common zero locus of the homogeneous quadratic polynomials vanishing on it.

(c) Suppose now that \( Q, Q' \subset \mathbb{P}^3 \) are two distinct quadric surfaces containing \( C \). Describe the intersection \( Q \cap Q' \).