**Qualifying Examination**

**HARVARD UNIVERSITY**

**Department of Mathematics**

**Tuesday, August 29, 2017 (Day 1)**

**Problem 1 (DG)**

a) Let $M$ denote a compact, smooth manifold. Show that $M$ can be imbedded in $\mathbb{R}^N$ if $N$ is sufficiently large.

b) Let $\pi: E \to M$ denote a smooth, finite rank real vector bundle. Show that $E$ is isomorphic to a subbundle of the product bundle $M \times \mathbb{R}^N$ if $N$ is sufficiently large.

**Problem 2 (T)**

Let $X$ denote the quotient space of $S^2$ that is obtained by identifying two distinct points.

a) Compute the homology groups $H_*(X; \mathbb{Z})$.

b) What is the universal covering space of $X$?

**Problem 3 (AN)**

Let $r = 2^{1/3}$. Let $K$ be the cubic number field $\mathbb{Q}(r)$, and $A$ its ring of integers. You may assume that $A = \mathbb{Z}[r]$.

a) Prove that if $p$ is an prime such that $p = 5 \mod(6)$, then the ideal $pA$ of $A$ factors as a product of two distinct prime ideals.

b) Find these two prime ideals with $p = 5$.

**Problem 4 (AG)**

a) Find all common solutions $(x, y)$ of the equations $f(x, y) = g(x, y) = 0$, where

$$f(x, y) = x^2y^2 - x^3y^2 \quad \text{and} \quad g(x, y) = x^2 - 2x + y^2 - 2y + 1.$$  

b) Let $\mathbb{C}[x, y]$ denote the ring of polynomials, and let $I = (f, g) \subset \mathbb{C}[x, y]$ denote the ideal generated by $f$ and $g$. Find the radical $\sqrt{I}$ of the ideal $I$. 
Problem 5 (RA)
Suppose that \((X, \mu)\) is a measure space with \(\mu(X)\) finite. Let \(p\) and \(q\) denote positive real numbers obeying \(1 \leq p < q \leq \infty\).

a) Prove that \(L^q(X)\) is a subset of \(L^p(X)\).

b) Let \(X = [0, 1]\) with \(\mu\) denoting Lebesgue measure. Give an example of a function that is in \(L^p\) but not in \(L^q\).

c) Give an example of a measure space \(X\) with \(\mu(X)\) infinite such that the reverse inclusion holds: Every \(L^p\) function is an \(L^q\) function.

Problem 6 (CA)
Prove that the infinite product \(f(z) = \prod_{n=1}^{\infty} ((1 - \frac{z}{n})e^{z/n})\) defines an analytic function on the whole of \(\mathbb{C}\) whose zeros are the positive integers.
Problem 1 (DG)
Let \( \mathcal{H}^2 \) denote the upper half plane in \( \mathbb{R}^2 \); the set \( \{(x,y) \in \mathbb{R}^2 : y > 0 \} \). Supposing that \( \alpha \) is a real number, equip \( \mathcal{H}^2 \) with the metric
\[
g_\alpha = \frac{dx^2 + dy^2}{y^\alpha}.
\]

a) Show that \( g_\alpha \) is not complete if \( \alpha \neq 2 \).

b) Let \( z = x + iy \). Fix a matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with real entries and determinant 1 (so an element in \( SL(2, \mathbb{R}) \)). Show that the map \( z \to \frac{az + b}{cz + d} \) maps \( \mathcal{H}^2 \) to itself as a diffeomorphism of \( \mathcal{H}^2 \), and that in so doing, it defines an isometry of the metric \( g_\alpha \).

Problem 2 (T)

a) Show that for all \( i \), the cohomology groups \( H^i(S^1 \times S^3; \mathbb{Z}) \) and \( H^i(S^1 \vee S^2 \vee S^3; \mathbb{Z}) \) are isomorphic.

b) Show that there does not exist a compact manifold that is homotopy equivalent to the wedge sum \( S^1 \vee S^2 \vee S^3 \).

Problem 3 (AN)

a) Show that the polynomial \( x^{11} - 1 \) has discriminant \( -11^{11} \).

b) Deduce that the polynomial \( C(x) = (x^{11} - 1)/(x - 1) \) in \( \mathbb{Q}[x] \) factors over \( \mathbb{Q}(\sqrt{-11}) \) as the product of two quintic polynomials, each with cyclic Galois group over \( \mathbb{Q}(\sqrt{-11}) \).

(You may use without proof the irreducibility of cyclotomic polynomials over \( \mathbb{Q} \).)
PROBLEM 4 (AG)

a) Define the Hilbert polynomial of a projective variety $X$ in $\mathbb{P}^n$.

b) Let $X \subseteq \mathbb{P}^4$ be a variety given as the intersection of a quadric and a cubic hypersurface with no common component. Show that the Hilbert polynomial of $X \subseteq \mathbb{P}^4$ is the polynomial $d \to 3d^2 + 2$.

PROBLEM 5 (RA)

Let $S^1$ denote the circle, $\mathbb{R}/(2\pi\mathbb{Z})$; and let $x \in S^1$ denote an irrational multiple of $2\pi$.

a) Suppose that $f: S^1 \to \mathbb{C}$ is a finite linear combination of functions from the set $\{e^{inx}: n \in \mathbb{Z}\}$. Prove the identity

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(kx) = \frac{1}{2\pi} \int_{0}^{2\pi} f(t) \, dt$$

b) Prove that this identity also holds for any continuous function $f$ on $S^1$ whose Fourier coefficients are absolutely summable. (This means that $f$ can be written as $f(t) = \sum_{n \in \mathbb{Z}} a_n e^{int}$ with $\sum_{n \in \mathbb{Z}} |a_n|$ being a convergent sequence.)

PROBLEM 6 (CA)

Supposing that $a > \sqrt{2}$, let $I(a) = \int_{0}^{2\pi} \frac{1}{a + \sin(\theta) + \cos(\theta)} \, d\theta$. Use contour integration to prove that $I(a) = \frac{2\pi}{\sqrt{a^2 - 2}}$. 
Qualifying Examination
HARVARD UNIVERSITY
Department of Mathematics
Thursday, August 31, 2017 (Day 3)

PROBLEM 1 (DG)
Suppose that $S$ is an embedded surface in $\mathbb{R}^3$.

a) Define the first and second fundamental forms of $S$.

b) Define the principle curvatures of $S$ at a given point.

c) Prove that if $S$ is compact, then the product of the principle curvatures of $S$ can not be negative at every point of $S$.

PROBLEM 2 (T)
Let $K$ denote the Klein bottle. It is obtained from a rectangle by making the edge identifications as indicated in the following picture:

Take the closed rectangle and identify the left blue side with the right head-to-head and tail-to-tail; and identify the upper red side with the lower red side head-to-head and tail-to-tail.

a) For which topological spaces $X$ does there exist a finite-sheeted covering map $X \to K$? (Hint: You may use the fact that if there exists an $n$-sheeted covering map from $X$ to a compact manifold $Y$, then $\chi(X) = n \chi(Y)$ with $\chi$ denoting the Euler characteristic.)

b) How many connected 2-sheeted covering spaces does $K$ have (up to automorphism)? How many of them are orientable?

PROBLEM 3 (AN)
Let $k$ denote a finite field of $q$ elements with $q > 2$, and let $G$ denote the group of permutations of $k$ that have the form $g_{a,b}: x \to ax + b$ with $a, b \in k$ and $a \neq 0$. 
a) Prove that two \textit{nonidentity} permutations \( g_{a,b} \) and \( g_{a',b'} \) are conjugate in \( G \) if and only if \( a = a' \). In particular, explain why this proves that \( G \) has \( q \) conjugacy classes.

b) Let \((V, \rho)\) denote the associated permutation representation over \( \mathbb{C} \), and \( V_0 \) the complement of the 1-dimensional trivial representation \( V^G \subset V \). Prove that \( \langle \chi_V, \chi_{V_0} \rangle = 2 \), and deduce that \( V_0 \) is irreducible.

c) How many distinct homomorphisms are there from \( k^* \) to \( \mathbb{C}^* \)? Show that any such homomorphism (call it \( \varphi \)) yields a 1-dimensional representation \( g_{a,b} \to \varphi(a) \) of \( G \). Explain why these representations together with \( V_0 \) give the complete set of isomorphism classes of irreducible representations of \( G \).

**Problem 4 (AG)**

Let \( C \) denote a smooth, projective curve, let \( K_c \) denote the canonical bundle of \( C \); and let \( \mathcal{L} \) denote a holomorphic line bundle on \( C \). The Riemann-Roch theorem says:

\[
h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}^{-1} \otimes K_C) = \deg(\mathcal{L}) + 1 - g(C)
\]

where \( \deg(\mathcal{L}) \) denotes the degree of \( \mathcal{L} \) and \( g(C) \) denotes the genus of \( C \). Use the Riemann-Roch theorem to prove that every curve \( C \) has a non-constant map to \( \mathbb{P}^1 \) of degree \( g(C) + 1 \) or less.

**Problem 5 (RA)**

Let \( g \) denote a smooth function on \( \mathbb{R}^3 \) with compact support. Let \( f \) denote the function given by the formula \( f(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} g(y) \, dy \).

a) Prove that the integral that defines \( f \) converges for each \( x \in \mathbb{R}^3 \) if \( g \) is a square integrable function on \( \mathbb{R}^3 \) with compact support.

b) Prove that \( f \) is differentiable and that the gradient of \( f \) is given by the formula

\[
\nabla f(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} (\nabla g)(y) \, dy.
\]

c) Prove that \( f \) obeys \( -\Delta f = g \) with \( \Delta \) denoting the Laplacian operator \( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \). (You are being asked to prove that the function \( x \rightarrow \frac{1}{4\pi |x-y|} \) is the Green's function for the Laplacian with pole at \( y \).)
Problem 6 (CA):
Suppose that $f$ is a $C^1$ map from an open disk in $\mathbb{R}^2$ to $\mathbb{R}^2$. Prove that the following are equivalent assertions (be careful not to give a circular proof):

a) $f$ obeys the Cauchy-Riemann equations.

b) When viewed as a $\mathbb{C}$-valued function on a disk in $\mathbb{C}$ (with $\mathbb{C}$ identified with $\mathbb{R}^2$ in the usual way) the function $f$ has a complex derivative in the following sense: Fix any point $z$ in the disk where $f$ is defined, and then a non-zero $h$ such that $z + h$ is in this disk. Then

$$\lim_{t \to 0} \frac{f(z+th) - f(z)}{th}$$

exists and it is independent of $h$. (The limit is taken along the line segment that is parametrized by the interval $[0, 1] \subset \mathbb{R}$ via the map $t \to z + th$.)