1. (ALGEBRA) Consider the algebra $M_2(k)$ of $2 \times 2$ matrices over a field $k$. Recall that an idempotent in an algebra is an element $e$ such that $e^2 = e$.

(a) Show that an idempotent $e \in M_2(k)$ different from 0 and 1 is conjugate to

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

by an element of $GL_2(k)$.

(b) Find the stabilizer in $GL_2(k)$ of $e_1 \in M_2(k)$ under the conjugation action.

(c) In case $k = \mathbb{F}_p$ is the prime field with $p$ elements, compute the number of idempotents in $M_2(k)$. (Count 0 and 1 in.)

2. (ALGEBRAIC GEOMETRY) (a) Find an everywhere regular differential $n$-form on the affine $n$-space $\mathbb{A}^n$.

(b) Prove that the canonical bundle of the projective $n$-dimensional space $\mathbb{P}^n$ is $O(-n-1)$.

3. (COMPLEX ANALYSIS) (Bol’s Theorem of 1949). Let $\hat{W}$ be a domain in $\mathbb{C}$ and $W$ be a relatively compact nonempty subdomain of $\hat{W}$. Let $\varepsilon > 0$ and $G_\varepsilon$ be the set of all $(a, b, c, d) \in \mathbb{C}$ such that $\max(|a - 1|, |b|, |c|, |d - 1|) < \varepsilon$. Assume that $cz + d \neq 0$ and $\frac{az + b}{cz + d} \in \hat{W}$ for $z \in W$ and $(a, b, c, d) \in G_\varepsilon$. Let $m \geq 2$ be an integer. Prove that there exists a positive integer $\ell$ (depending on $m$) with the property that for any holomorphic function $\varphi$ on $\hat{W}$ such that

$$\varphi(z) = \varphi\left(\frac{az + b}{cz + d}\right) \frac{(cz + d)^{2m}}{(ad - bc)^m}$$

for $z \in W$ and $(a, b, c, d) \in G_\varepsilon$, the $\ell$-th derivative $\psi(z) = \varphi^{(\ell)}(z)$ of $\varphi(z)$ on $\hat{W}$ satisfies the equation

$$\psi(z) = \psi\left(\frac{az + b}{cz + d}\right) \frac{(ad - bc)^{\ell-m}}{(cz + d)^{2(\ell-m)}}$$

for $z \in W$ and $(a, b, c, d) \in G_\varepsilon$. Express $\ell$ in terms of $m$.

*Hint:* Use Cauchy’s integral formula for derivatives.
4. (ALGEBRAIC TOPOLOGY) (a) Show that the Euler characteristic of any contractible space is 1.

(b) Let $B$ be a connected CW complex made of finitely many cells so that its Euler characteristic is defined. Let $E \to B$ be a covering map whose fibers are discrete, finite sets of cardinality $N$. Show the Euler characteristic of $E$ is $N$ times the Euler characteristic of $B$.

(c) Let $G$ be a finite group with cardinality $> 2$. Show that $BG$ (the classifying space of $G$) cannot have homology groups whose direct sum has finite rank.

5. (DIFFERENTIAL GEOMETRY) Let $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ be the upper half plane. Let $g$ be the Riemannian metric on $H$ given by

$$g = \frac{(dx)^2 + (dy)^2}{y^2}.$$  

$(H, g)$ is known as the half-plane model of the hyperbolic plane.

(a) Let $\gamma(\theta) = (\cos \theta, \sin \theta)$ and $\eta(\theta) = (\cos \theta + 1, \sin \theta)$ for $\theta \in (0, \pi)$ be two paths in $H$. Compute the angle $A$ at their intersection point shown in Figure 1, measured by the metric $g$.

(b) By computing the Levi-Civita connection

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{2} \Gamma_{ij}^{k} \frac{\partial}{\partial x_k}$$

of $g$ or otherwise (where $(x_1, x_2) = (x, y)$), show that the path $\gamma$, after arclength reparametrization, is a geodesic with respect to the metric $g$. 

Figure 1: Angle $A$ between the two curves $\gamma$ and $\eta$ in the upper half plane $H$. 

(b) By computing the Levi-Civita connection
6. (Real Analysis) For any positive integer $n$ let $M_n$ be a positive number such that the series $\sum_{n=1}^{\infty} M_n$ of positive numbers is convergent and its limit is $M$. Let $a < b$ be real numbers and $f_n(x)$ be a real-valued continuous function on $[a, b]$ for any positive integer $n$ such that its derivative $f'_n(x)$ exists for every $a < x < b$ with $|f'_n(x)| \leq M_n$ for $a < x < b$. Assume that the series $\sum_{n=1}^{\infty} f_n(a)$ of real numbers converges. Prove that

(a) the series $\sum_{n=1}^{\infty} f_n(x)$ converges to some real-valued function $f(x)$ for every $a \leq x \leq b$,

(b) $f'(x)$ exists for every $a < x < b$, and

(c) $|f'(x)| \leq M$ for $a < x < b$.

Hint for (b): For fixed $x \in (a, b)$ consider the series of functions

$$\sum_{n=1}^{\infty} \frac{f_n(y) - f_n(x)}{y - x}$$

of the variable $y$ and its uniform convergence.
1. (Algebra) Find all the field automorphisms of the real numbers $\mathbb{R}$.

*Hint:* Show that any automorphism maps a positive number to a positive number, and deduce from this that it is continuous.

2. (Algebraic Geometry) What is the maximum number of ramification points that a mapping of finite degree from one smooth projective curve over $\mathbb{C}$ of genus 1 to another (smooth projective curve of genus 1) can have? Give an explanation for your answer.

3. (Complex Analysis) Let $\omega$ and $\eta$ be two complex numbers such that $\text{Im} \left( \frac{\omega}{\eta} \right) > 0$. Let $G$ be the closed parallelogram consisting of all $z \in \mathbb{C}$ such that $z = \lambda \omega + \rho \eta$ for some $0 \leq \lambda, \rho \leq 1$. Let $\partial G$ be the boundary of $G$ and let $G^0 = G - \partial G$ be the interior of $G$. Let $P_1, \cdots, P_k, Q_1, \cdots, Q_\ell$ be points in $G^0$ and let $m_1, \cdots, m_k, n_1, \cdots, n_\ell$ be positive integers. Let $f$ be a function on $G$ such that

$$ f(z) \prod_{j=1}^{\ell} (z - Q_j)^{n_j} \prod_{p=1}^{k} (z - P_p)^{m_p} $$

is continuous and nowhere zero on $G$ and is holomorphic on $G^0$. Let $\varphi(z)$ and $\psi(z)$ be two polynomials on $\mathbb{C}$. Assume that $f(z + \omega) = e^{\varphi(z)} f(z)$ if both $z$ and $z + \omega$ are in $G$. Assume also that $f(z + \eta) = e^{\psi(z)} f(z)$ if both $z$ and $z + \eta$ are in $G$. Express $\sum_{p=1}^{k} m_p - \sum_{j=1}^{\ell} n_j$ in terms of $\omega$ and $\eta$ and the coefficients of $\varphi(z)$ and $\psi(z)$.

4. (Algebraic Topology) (a) Fix a basis for $H_1$ of the two-torus (with integer coefficients). Show that for every element $x \in SL(2, \mathbb{Z})$, there is an automorphism of the two-torus such that the induced map on $H_1$ acts by $x$.

*Hint:* $SL(2, \mathbb{Z})$ also acts on the universal cover of the torus.

(b) Fix an embedding $j : D^2 \times S^1 \to S^3$. Remove its interior from $S^3$ to obtain a manifold $X$ with boundary $T^2$. Let $f$ be an automorphism of the two-torus and consider the glued space

$$ X_f := (D^2 \times S^1) \cup_f X. $$

If $X$ is homotopy equivalent to $D^2 \times S^1$, compute the homology groups of $X_f$. 

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5. **(Differential Geometry)** Let \( M = U(n)/O(n) \) for \( n \geq 1 \), where \( U(n) \) is the group of \( n \times n \) unitary matrices and \( O(n) \) is the group of \( n \times n \) orthogonal matrices. \( M \) is a real manifold called the *Lagrangian Grassmannian*.

(a) Compute and state the dimension of \( M \).

(b) Construct a Riemannian metric which is invariant under the left action of \( U(n) \) on \( M \).

(c) Let \( \nabla \) be the corresponding Levi-Civita connection on the tangent bundle \( TM \), and \( X, Y, Z \) be any \( U(n) \)-invariant vector fields on \( M \). Using the given identity (which you are not required to prove)

\[
\nabla_X Y = \frac{1}{2}[X, Y],
\]

show that the Riemannian curvature tensor \( R \) of \( \nabla \) satisfies the formula

\[
R(X, Y)Z = \frac{1}{4}[Z, [X, Y]].
\]

6. **(Real Analysis)** Show that there is no function \( f : \mathbb{R} \to \mathbb{R} \) whose set of continuous points is precisely the set \( \mathbb{Q} \) of all rational numbers.
1. (Algebra) Consider the function fields $K = \mathbb{C}(x)$ and $L = \mathbb{C}(y)$ of one variable, and regard $L$ as a finite extension of $K$ via the $\mathbb{C}$-algebra inclusion $x \mapsto -\frac{(y^5 - 1)^2}{4y^5}$

Show that the extension $L/K$ is Galois and determine its Galois group.

2. (Algebraic Geometry) Is every smooth projective curve of genus 0 defined over the field of complex numbers isomorphic to a conic in the projective plane? Give an explanation for your answer.

3. (Complex Analysis) Let $f(z) = z + e^{-z}$ for $z \in \mathbb{C}$ and let $\lambda \in \mathbb{R}$, $\lambda > 1$. Prove or disprove the statement that $f(z)$ takes the value $\lambda$ exactly once in the open right half-plane $H_r = \{z \in \mathbb{C} : \text{Re } z > 0\}$.

4. (Algebraic Topology) (a) Let $X$ and $Y$ be locally contractible, connected spaces with fixed basepoints. Let $X \vee Y$ be the wedge sum at the basepoints. Show that $\pi_1(X \vee Y)$ is the free product of $\pi_1X$ with $\pi_1Y$.

(b) Show that $\pi_1(X \times Y)$ is the direct product of $\pi_1X$ with $\pi_1Y$.

(c) Note the canonical inclusion $f : X \vee Y \to X \times Y$. Assume that $X$ and $Y$ have abelian fundamental groups. Show that the map $f_*$ on fundamental groups exhibits $\pi_1(X \times Y)$ as the abelianization of $\pi_1(X \vee Y)$.

Hint: The Hurewicz map is natural.

5. (Differential Geometry) (a) Let $S^1 = \mathbb{R}/\mathbb{Z}$ be a circle and consider the connection

$$\nabla := d + \pi \sqrt{-1} d\theta$$

defined on the trivial complex line bundle over $S^1$, where $\theta$ is the standard coordinate on $S^1 = \mathbb{R}/\mathbb{Z}$ descended from $\mathbb{R}$. By solving the differential equation for flat sections $f(\theta)$

$$\nabla f = df + \pi \sqrt{-1} f d\theta = 0$$

or otherwise, show that there does not exist global flat sections with respect to $\nabla$ over $S^1$. 


(b) Let $T = V/\Lambda$ be a torus, where $\Lambda$ is a lattice and $V = \Lambda \otimes \mathbb{R}$ is the real vector space containing $\Lambda$. Let $L$ be the trivial complex line bundle equipped with the standard Hermitian metric. By identifying flat $U(1)$ connections with $U(1)$ representations of the fundamental group $\pi_1(T)$ or otherwise, show that the space of flat unitary connections on $L$ is the dual torus $T^* = V^*/\Lambda^*$, where $\Lambda^* := \text{Hom}(\Lambda, \mathbb{Z})$ is the dual lattice and $V^* := \text{Hom}(V, \mathbb{R})$ is the dual vector space.

6. (Real Analysis) (Fundamental Solutions of Linear Partial Differential Equations with Constant Coefficients). Let $\Omega$ be an open interval $(-M, M)$ in $\mathbb{R}$ with $M > 0$. Let $n$ be a positive integer and $L = \sum_{\nu=0}^{n} a_{\nu} \frac{d^\nu}{dx^\nu}$ be a linear differential operator of order $n$ on $\mathbb{R}$ with constant coefficients, where the coefficients $a_0, \cdots, a_{n-1}, a_n \neq 0$ are complex numbers and $x$ is the coordinate of $\mathbb{R}$. Let $L^* = \sum_{\nu=0}^{n} (-1)^n \overline{a_{\nu}} \frac{d^n}{dx^n}$. Prove, by using Plancherel’s identity, that there exists a constant $c > 0$ which depends only on $M$ and $a_n$ and is independent of $a_0, a_1, \cdots, a_{n-1}$ such that for any $f \in L^2(\Omega)$ a weak solution $u$ of $Lu = f$ exists with $\|u\|_{L^2(\Omega)} \leq c \|f\|_{L^2(\Omega)}$. Give one explicit expression for $c$ as a function of $M$ and $a_n$.

Hint: A weak solution $u$ of $Lu = f$ means that $(f, \psi)_{L^2(\Omega)} = (u, L^* \psi)_{L^2(\Omega)}$ for every infinitely differentiable function $\psi$ on $\Omega$ with compact support. For the solution of this problem you can consider as known and given the following three statements.

(I) If there exists a positive number $c > 0$ such that $\|\psi\|_{L^2(\Omega)} \leq c \|L^* \psi\|_{L^2(\Omega)}$ for all infinitely differentiable complex-valued functions $\psi$ on $\Omega$ with compact support, then for any $f \in L^2(\Omega)$ a weak solution $u$ of $Lu = f$ exists with $\|u\|_{L^2(\Omega)} \leq c \|f\|_{L^2(\Omega)}$.

(II) Let $P(z) = z^m + \sum_{k=0}^{m-1} b_k z^k$ be a polynomial with leading coefficient 1. If $F$ is a holomorphic function on $\mathbb{C}$, then

$$|F(0)|^2 \leq \frac{1}{2\pi} \int_{\theta=0}^{2\pi} |P(e^{i\theta}) F(e^{i\theta})|^2 d\theta.$$ 

(III) For an $L^2$ function $f$ on $\mathbb{R}$ which is zero outside $\Omega = (-M, M)$ its Fourier transform

$$\hat{f}(\xi) = \int_{-M}^{M} f(x)e^{-2\pi i x \xi} dx.$$
as a function of $\xi \in \mathbb{R}$ can be extended to a holomorphic function

$$
\hat{f}(\xi + i\eta) = \int_{-M}^{M} f(x) e^{-2\pi i x (\xi + i\eta)} dx
$$
on $\mathbb{C}$ as a function of $\xi + i\eta$. 

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