1. (CA) Evaluate
\[ \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} \, dx \]

**Solution.** Let \( C \) be the curve on the complex plane from \(-\infty\) to \(+\infty\), which is along the real line for most part but gets around the origin by going upwards (clockwise). We are integrating
\[
\int_C \frac{\sin^2 z}{z^2} \, dz = \int_C \left( \frac{2 - e^{2iz} - e^{-2iz}}{4z^2} \right) \, dz = \int_C \frac{1 - e^{2iz}}{4z^2} \, dz + \int_C \frac{1 - e^{-2iz}}{4z^2} \, dz.
\]

Let \( C' \) be the curve from \(-\infty\) to \(+\infty\), along the real line for most part but now goes downwards around the origin. Then
\[
\int_C \frac{1 - e^{-2iz}}{4z^2} \, dz - \int_{C'} \frac{1 - e^{-2iz}}{4z^2} \, dz = -2\pi i \cdot \text{Res}_{z=0} \left( \frac{1 - e^{-2iz}}{4z^2} \right) = \pi.
\]

As \( 1 - e^{2iz} \) is bounded when \( \text{Im}(z) \geq 0 \), \( \int_C \frac{1 - e^{2iz}}{4z^2} \, dz = 0 \) as we can push the integral up to infinity. Similarly \( \int_{C'} \frac{1 - e^{-2iz}}{4z^2} \, dz = 0 \). This shows the original integral has value \( \pi \).

2. (A) Let \( b \) be any integer with \( (7, b) = 1 \) and consider the polynomial
\[ f_b(x) = x^3 - 21x + 35b. \]

(a) Show that \( f_b \) is irreducible over \( \mathbb{Q} \).

(b) Let \( P \) denote the set of \( b \in \mathbb{Z} \) such that \( (7, b) = 1 \) and the Galois group of \( f_b \) is the alternating group \( A_3 \). Find \( P \).

**Solution.**

(a) This follows from the Eisenstein criterion on the prime 7.

(b) From (a), the Galois group is \( A_3 \) if the discriminant is a square (in \( \mathbb{Q} \)), and \( S_3 \) otherwise. The discriminant of \( x^3 + ax + b \) is \(-4a^3 - 27b^2\), and the discriminant of \( f_b \) is \( 4 \cdot 21^3 - 27 \cdot 35^2 \cdot b^2 = 3^37^2(28 - 25b) \). Thus we’re looking for all \( b \) such that \( 3(28 - 25b) \) is a square. Such square must be divisible by 9 and are congruent to 9 modulo 25, hence of the form \( (75n \pm 3)^2 \), i.e. \( 3(28 - 25b) = 5625n^2 \pm 450n + 9 \Leftrightarrow b = -75n^2 \pm 6n + 1 \). Thus \( P = \{ -75n^2 + 6n + 1 \mid n \in \mathbb{Z} \} \).
3. (T) Let $X$ be the Klein bottle, obtained from the square $I^2 = \{(x,y) : 0 \leq x, y \leq 1\} \subset \mathbb{R}^2$ by the equivalence relation $(0,y) \sim (1,y)$ and $(x,0) \sim (1-x,1)$.

(a) Compute the homology groups $H_n(X,\mathbb{Z})$.

(b) Compute the homology groups $H_n(X,\mathbb{Z}/2)$.

(c) Compute the homology groups $H_n(X \times X,\mathbb{Z}/2)$.

Solution. $X$ has the following cellular decomposition: the square $F$, the edges $E_1 = \{0\} \times [0,1]$ and $E_2 = [0,1] \times \{0\}$, and the vertex $V = (0,0)$. We have $\delta F = 2E_1$ and $\delta E_1 = \delta E_2 = \delta V = 0$.

(a) $H_2(X,\mathbb{Z}) = \{c \cdot F | \delta(c \cdot F) = 0\} = 0$. As all other boundary maps are zero, $H_1(X,\mathbb{Z}) = (\mathbb{Z}E_1 + \mathbb{Z}E_2)/2 \mathbb{Z}E_1 \cong (\mathbb{Z}/2) \oplus \mathbb{Z}$ and $H_0(X,\mathbb{Z}) = \mathbb{Z}$.

(b) All boundary maps are zero in $\mathbb{Z}/2$-coefficient. Thus $H_2(X,\mathbb{Z}/2) = \mathbb{Z}/2$, $H_1(X,\mathbb{Z}/2) = (\mathbb{Z}/2)^2$ and $H_0(X,\mathbb{Z}/2) = \mathbb{Z}/2$.

(c) $\mathbb{Z}/2$ may be seen as a field. Thus $H^i(X,\mathbb{Z}/2) = H_i(X,\mathbb{Z}/2)$ for any $i$ and by the Kunneth formula $H^*(X \times X,\mathbb{Z}/2) = H^*(X,\mathbb{Z}/2)^{\otimes 2}$. Explicitly

$$H^i(X \times X,\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & i = 0, 4 \\ (\mathbb{Z}/2)^4 & i = 1, 3 \\ (\mathbb{Z}/2)^6 & i = 2 \\ 0 & \text{else} \end{cases}$$

4. (RA) Let $f$ be a Lebesgue integrable function on the closed interval $[0,1] \subset \mathbb{R}$.

(a) Suppose $g$ is a continuous function on $[0,1]$ such that the integral of $|f - g|$ is less than $\epsilon^2$. Prove that the set where $|f - g| > \epsilon$ has measure less than $\epsilon$.

(b) Show that for every $\epsilon > 0$, there is a continuous function $g$ on $[0,1]$ such that the integral of $|f - g|$ is less than $\epsilon^2$.

Solution.

(a) This is obvious.

(b) We have to prove that continuous functions are dense As $f$ is Lebesgue integrable, $f$ can be $L^1$-approximated by step functions, i.e. for any $\delta > 0$, there exist real numbers $c_1, \ldots, c_n$ and measurable sets $E_1, \ldots, E_n \subset [0,1]$ such that the integral of $|f - c_1 \chi_{E_1} - \ldots - c_n \chi_{E_n}|$ is smaller than $\delta$, where we denote by $\chi_E$ the characteristic function of $E$. By picking small enough $\delta$ and replace $f$ by $c_1 \chi_{E_1} + \ldots + c_n \chi_{E_n}$, it suffices to prove that for any $\epsilon > 0$ and any characteristic function $\chi_E$ of a measurable set $E \subset [0,1]$, there is a continuous function $g_E$ such that the integral of $|g_E - \chi_E|$ is smaller than $\epsilon$
As the Lebesgue measure is inner and outer regular, we may find compact $K$ and open $U$ such that $K \subset E \subset U$ and the measure of $U - K$ is arbitrarily small. Urysohn lemma now gives us a continuous function that is 1 on $K$, 0 on $[0, 1] - U$ and between 0 and 1 in $U - K$. This gives the required function $g_E$.

5. (DG) Let $v$ denote a vector field on a smooth manifold $M$ and let $p \in M$ be a point. An integral curve of $v$ through $p$ is a smooth map $\gamma : U \to M$ from a neighborhood $U$ of $0 \in \mathbb{R}$ to $M$ such that $\gamma(0) = p$ and the differential $d\gamma$ carries the tangent vector $\partial/\partial t$ to $v(\gamma(t))$ for all $t \in U$.

(a) Prove that for any $p \in M$ there is an integral curve of $v$ through $p$.

(b) Prove that any two integral curves of $v$ through any given point $p$ agree on some neighborhood of $0 \in \mathbb{R}$.

(c) A complete integral curve of $v$ through $p$ is one whose associated map has domain the whole of $\mathbb{R}$. Give an example of a nowhere zero vector field on $\mathbb{R}^2$ that has a complete integral curve through any given point. Then, give an example of a nowhere zero vector field on $\mathbb{R}^2$ and a point which has no complete integral curve through it.

Solution. Pick a local chart of the manifold $M$ at the considered point $p$. The chart may be seen as a neighborhood of a point $p \in \mathbb{R}^n$, and the vector field $v$ is also given on the neighborhood. To give an integral curve through a point $p$ is then to solve the ordinary differential equation (system) $x'(t) = v(x(t))$ and $x(0) = p$. As $v$ is smooth and thus $C^1$, (a) and (b) follows from the (local) existence and uniqueness of solutions for ordinary differential equations.

For (c), constant vector field $v(x, y) = (1, 0)$ on $\mathbb{R}^2$ gives the first required example. Horizontal curves parametrized by arc length are all possible integral curves. For the second required example, we may consider $v(x, y) = (x^2, 0)$. A integral curve with respect to such a vector field is a solution to the ODE $x'(t) = x(t)^2$. Such a solution is of the form $\frac{1}{t-a}$, and always blows up in finite time (either forward or backward), i.e. there is no complete integral curve for this vector field.

6. (AG) Show that a general hypersurface $X \subset \mathbb{P}^n$ of degree $d > 2n - 3$ contains no lines $L \subset \mathbb{P}^n$.

Solution. A hypersurface in $\mathbb{P}^n$ of degree $d$ is given by a homogeneous polynomial in $n + 1$ variable $x_0, x_1, ..., x_n$ of degree $d$ up to a constant. There are $k(d, n) = \binom{d + n}{n}$ such monomials, and thus the space of such polynomials is $\mathbb{P}^{k(d,n)-1}$. After a change of coordinate a line may be expressed as $x_2 = x_3 = ... = x_n = 0$. A hypersurface that contains this line then corresponds to a polynomial with no $x_0^d, x_0^{d-1}x_1, ..., x_1^d$ terms, which constitutes a codimension $d + 1$ subplane. On the other hand, the grassmannian of lines
is a variety of dimension $2 \cdot ((n + 1) - 2) = 2n - 2 < d + 1$. This proves the assertion.

To be more rigorous, let $G$ be the grassmannian of lines in $\mathbb{P}^n$, $H \cong \mathbb{P}^{k(d,n)-1}$ the space of hypersurfaces of degree $d$ in $\mathbb{P}^n$. We may consider

$$X = \{(l, S) \mid l \in G, S \in H \text{ such that } l \subset S\}.$$ 

Then what we have learned is that $G$ has dimension $2n - 2$ and the fiber of the projection map $X \to G$ has dimension $d + 1$ less than the dimension of $H$. Thus the dimension of $X$ is the sum of the dimension of $G$ and the dimension of the fiber $H$, which is smaller than the dimension of $H$ exactly when $d > 2n - 3$. It follows that the projection $X \to H$ cannot be surjective, which is the assertion to be proved.
1. (T) If $M_g$ denotes the closed orientable surface of genus $g$, show that continuous maps $M_g \rightarrow M_h$ of degree 1 exist if and only if $g \geq h$.

**Solution.** A closed orientable surface of genus $g \geq 1$ may be described as a polygon of $4g$ edges, some pairs identified in a certain way. In particular, all vertices are identified together under this identification. For $g > h$, by further identify $4g - 4h$ edges to the point, we can construct a map from $M_g$ to $M_h$ which is a homeomorphism on the interior of the polygon (2-cell). Since the 2-cell is the generator of $H_2(\cdot, \mathbb{Z})$, the map constructed has degree 1.

If $g < h$, then any map $f : M_g \rightarrow M_h$ induces $f^* : H^1(M_h, \mathbb{Z}) \rightarrow H^1(M_g, \mathbb{Z})$, which cannot be injective since the former is a free abelian group of rank $2h$ and the latter has rank $2g$. Pick $0 \neq \alpha \in H^1(M_h, \mathbb{Z})$ such that $f^*(\alpha) = 0$, there always exists $\beta$ with $\alpha, \beta \neq 0 \in H^2(M_h, \mathbb{Z})$. However $f^*(\alpha, \beta) = f^*(\alpha), f^*(\beta) = 0$, and thus $f$ must has degree 0.

2. (RA) Let $f \in C(S^1)$ be continuous function with a continuous first derivative $f'(x)$. Let $\{a_n\}$ be the Fourier coefficient of $f$. Prove that $\sum_n |a_n| < \infty$.

**Solution.** $f'$ has $n$-th Fourier coefficient equal to $na_n$. We thus have

$$\|f'\|_{L^2}^2 = \sum_n n^2 a_n^2 < \infty.$$

Then $(\sum_n |a_n|)^2 \leq (\sum_n n^2 a_n^2)(\sum_n 1/n^2) < \infty$ by Cauchy’s inequality.

3. (DG) Let $S \subset \mathbb{R}^3$ be the surface given as a graph

$$z = ax^2 + 2bxy + cy^2$$

where $a$, $b$ and $c$ are constants.

(a) Give a formula for the curvature at $(x, y, z) = (0, 0, 0)$ of the induced Riemannian metric on $S$.

(b) Give a formula for the second fundamental form at $(x, y, z) = (0, 0, 0)$.

(c) Give necessary and sufficient conditions on the constants $a$, $b$ and $c$ that any curve in $S$ whose image under projection to the $(x, y)$-plane is a straight line through $(0, 0)$ is a geodesic on $S$.

**Solution.** Let normal vectors to the surface may be expressed as $n(x, y, z) = l(x, y, z) \cdot (ax + by, bx + cy, -1)$, where $l(x, y, z)$ is the inverse of the length
of the vector. Note that \( l(x, y, z) = 0 \) and the first derivative of \( l(x, y, z) \) is zero at \((0, 0, 0) \in S\). When we compute the second fundamental form we only have to compute the first derivative of \( n(x, y, z) \). Therefore to compute the second fundamental form at \((0, 0, 0)\) we can treat \( l(x, y, z) \equiv 1 \) and the second fundamental form is thus

\[
\begin{pmatrix}
\frac{\partial}{\partial x}(ax + by) \\
\frac{\partial}{\partial y}(ax + by) \\
\frac{\partial}{\partial x}(bx + cy) \\
\frac{\partial}{\partial y}(bx + cy)
\end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}
\]

The curvature of the surface at \((0, 0, 0)\) is the determinant at the point, i.e. \( ac - b^2 \). For any curve whose projection to the \((x, y)\)-plane is a straight line through the point to be a geodesic, the corresponding vector has to be an eigenvector of the matrix. Thus it is necessary that \( a = c, b = 0 \), i.e. the matrix being a multiple of the identity matrix for that to happen. On the other hand, when \( a = c, b = 0 \), the surface is radially symmetric and thus all such curves must be geodesics.

4. (AG) Let \( V \) and \( W \) be complex vector spaces of dimensions \( m \) and \( n \) respectively and \( A \subset V \) a subspace of dimension \( l \). Let \( \mathbb{P}\text{Hom}(V, W) \cong \mathbb{P}^{mn-1} \) be the projective space of nonzero linear maps \( \phi : V \to W \) mod scalars, and for any integer \( k \leq l \) let

\[
\Psi_k = \{ \phi : V \to W : \text{rank}(\phi|_A) \leq k \} \subset \mathbb{P}^{mn-1}.
\]

Show that \( \Psi_k \) is an irreducible subvariety of \( \mathbb{P}^{mn-1} \), and find its dimension.

**Solution.** An \( n \times m \) matrix of rank \( \leq k \) can be decomposed into the product of an \( n \times k \) matrix and a \( k \times m \) matrix. Let \( X \cong \mathbb{P}^{nk-1} \) and \( Y \cong \mathbb{P}^{km-1} \) be the space of nonzero such matrices mod scalars. Then we have a surjection \( X \times Y \to \Psi_k \) by the multiplication map. This shows that \( \Psi_k \), as the image of the complete irreducible variety \( X \times Y \), is irreducible and closed in \( \mathbb{P}^{mn-1} \).

When a matrix has rank exactly \( k \), the decomposition has a \( GL(k) \) freedom of choice, i.e. each fiber of this map over a point in \( \Psi_k - \Psi_{k-1} \) has dimension \( k^2 - 1 \). As \( \Psi_{k-1} \) is closed in irreducible \( \Psi_k \), \( \dim \Psi_k = \dim(\Psi_k - \Psi_{k-1}) = \dim(X \times Y) - (k^2 - 1) = (nk - 1) + (mk - 1) - (k^2 - 1) = k(n + m - k) - 1 \).

(We used the fact that \( k \leq l \), in which case \( \Psi_k - \Psi_{k-1} \) is obviously non-empty.)

5. (CA) Find a conformal map from the region

\[
\Omega = \{ z : |z - 1| > 1 \text{ and } |z - 2| < 2 \} \subset \mathbb{C}
\]

between the two circles \(|z - 1| = 1\) and \(|z - 2| = 2\) onto the upper-half plane.

**Solution.** Let \( S = \left\{ \frac{1}{4} \leq \text{Re}(z) \leq \frac{1}{2} \right\} \subset \mathbb{C} \). Then we have \( \Omega \cong S \) by \( z \mapsto \frac{1}{z} \) and \( S \cong \text{upper-half plane} \) by \( z \mapsto e^{2\pi i(z-\frac{1}{4})} \).
6. (A) Let $G$ be a finite group with an automorphism $\sigma : G \rightarrow G$. If $\sigma^2 = id$ and the only element fixed by $\sigma$ is the identity of $G$, show that $G$ is abelian.

Solution. Define $\tau(x) := \sigma(x)x^{-1}$, then by assumption $\tau(x) \neq e$, $\forall x \neq e$. For any $x \neq x'$, $\tau(x)\tau(x')^{-1} = \sigma(x)x^{-1}x'\sigma(x')^{-1}$ is conjugate to $\sigma(x')^{-1}\sigma(x)x^{-1}x' = \tau(x^{-1}x) \neq e$, i.e. $\tau(x) \neq \tau(x')$. Thus $\tau : G \rightarrow G$ is a surjective function. But we have $\sigma(\tau(x)) = \sigma(x)\sigma(x)^{-1} = \tau(x)^{-1}$, hence $\sigma(x) = x^{-1}$ and $G$ is abelian.
1. (DG) Let $D \subset \mathbb{R}^2$ be the closed unit disk, with boundary $\partial D \cong S^1$. For any smooth map $\gamma : D \to \mathbb{R}^2$, let $A(\gamma)$ denote the integral over $D$ of the pull-back $\gamma^*(dx \wedge dy)$ of the area 2-form $dx \wedge dy$ on $\mathbb{R}^2$.

(a) Prove that $A(\gamma) = A(\gamma')$ if $\gamma = \gamma'$ on the boundary of $D$.

(b) Let $\alpha : \partial D \to \mathbb{R}^2$ denote a smooth map, and let $\gamma : D \to \mathbb{R}^2$ denote a smooth map such that $\gamma|_{\partial D} = \alpha$. Give an expression for $A(\gamma)$ as an integral over $\partial D$ of a function that is expressed only in terms of $\alpha$ and its derivatives to various orders.

(c) Give an example of a map $\gamma$ such that $\gamma^*(dx \wedge dy)$ is a positive multiple of $dx \wedge dy$ at some points and a negative multiple at others.

Solution. Consider the differential $\omega = ydx$ on $\mathbb{R}^2$, so $d\omega = dx \wedge dy$. We have $\gamma^*(d\omega) = d\gamma^*(\omega)$. Thus if $\gamma|_{\partial D} = \alpha$, the by Stoke’s theorem

$$
\int_D \gamma^*(dx \wedge dy) = \int_D d\gamma^*(\omega) = \int_{\partial D} \alpha^*(\omega).
$$

and depends only on $\alpha$ instead of $\gamma$. This finishes both (a) and (b).

For (c), one take for example $\gamma(x,y) = (x^2, y)$, then $\gamma^*(dx \wedge dy) = 2x(dx \wedge dy)$.

2. (T) Compute the fundamental group of the space $X$ obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

Solution. The space $X$ is $S^1 \times Y$, where $Y$ is obtained from two circle $S^1$ by identifying a point $x_0 \in S^1$ with the corresponding point on the other circle. Thus $\pi_1(X) = \pi_1(S^1) \times \pi_1(Y)$, the product of $\mathbb{Z}$ and a free group on two generators.

3. (CA) Let $u$ be a positive harmonic function on $\mathbb{C}$. Show that $u$ is a constant.

Solution. There exists a holomorphic function $f$ on $\mathbb{C}$ such that $u$ is the real part of it. Then $e^{-f}$ has image in the unit disk. By Liouville’s theorem $e^{-f}$ must be a constant, hence so is $u$.

4. (A) Let $R = \mathbb{Z}[\sqrt{-5}]$. Express the ideal $(6) = 6R \subset R$ as a product of prime ideals in $R$.

Solution. $(6) = (2)(3)$ and $(2) = (2, 1 + \sqrt{-5})^2$, $(3) = (3, 1 + \sqrt{-5})(3, 1 - \sqrt{-5})$. The final resulting ideals are prime because their indices (to $R$) are prime numbers.
5. (AG) Let $Q \subset \mathbb{P}^5$ be a smooth quadric hypersurface, and $L \subset Q$ a line. Show that there are exactly two 2-planes $\Lambda \cong \mathbb{P}^2 \subset \mathbb{P}^5$ contained in $Q$ and containing $L$.

Solution. By a linear change of coordinate we may assume the line is $x_2 = x_3 = x_4 = x_5 = 0$, where $x_0, \ldots, x_5$ are coordinates of the projective space $\mathbb{P}^5$. Then the degree two homogeneous polynomial defining $Q$ may be written as $F = f_0(x_2, x_3, x_4, x_5) x_0 + f_1(x_2, x_3, x_4, x_5) x_1 + q(x_2, x_3, x_4, x_5)$, where neither $f_0$ nor $f_1$ are a constant multiple of the other since $\frac{\partial F}{\partial x_0}$ and $\frac{\partial F}{\partial x_1}$ have to be independent for $Q$ to be smooth. We may thus arrange another change of coordinate among $x_2, x_3, x_4, x_5$ so that $f_0 = x_2, f_1 = x_3$. The $F = x_0 x_2 + x_1 x_3 + q(x_2, x_3, x_4, x_5)$.

Any plane that lies within $Q$ and contains $L$ is then of the form $(x_0 = x_1 = 0, ax_4 + bx_5 = 0)$, where $ax_4 + bx_5$ is nontrivial and divides $q(0, 0, x_4, x_5)$. Note $\frac{\partial F}{\partial x_0} = x_2$ and $\frac{\partial F}{\partial x_1} = x_3$. For $Q$ to be smooth we need $\frac{\partial F}{\partial x_4} q(0, 0, x_4, x_5)$ and $\frac{\partial F}{\partial x_5} q(0, 0, x_4, x_5)$ have to be independent, which is equivalent to $q(0, 0, x_4, x_5)$ is non-degenerate, in which case it has two linear divisors.

6. (RA) Let $C^\infty$ denote the space of smooth, real-valued functions on the closed interval $I = [0, 1]$. Let $\mathbb{H}$ denote the completion of $C^\infty$ using the norm whose square is the functional

$$f \mapsto \int_I \left( \left( \frac{df}{dt} \right)^2 + f^2 \right) dt.$$

(a) Prove that the map of $C^\infty$ to itself given by $f \mapsto T(f)$ with

$$T(f)(t) = \int_0^t f(s) ds$$

extends to give a bounded map from $\mathbb{H}$ to $\mathbb{H}$, and prove that the norm of $T$ is 1. (Remark: Its norm is actually not 1)

(b) Prove that $T$ is a compact mapping from $\mathbb{H}$ to $\mathbb{H}$.

(c) Let $C^{1/2}$ be the Banach space obtained by completing $C^\infty$ using the norm given by

$$f \mapsto \sup_{t \neq t'} \frac{|f(t) - f(t')|}{|t - t'|^{1/2}} + \sup_t |f(t)|.$$

Prove that the inclusion of $C^\infty$ into $\mathbb{H}$ and into $C^{1/2}$ extends to give a bounded, linear map from $\mathbb{H}$ to $C^{1/2}$.

(d) Give an example of a sequence in $\mathbb{H}$ such that all elements have norm 1 and such that there are no convergent subsequences in $C^{1/2}$.

Solution.
(a) To prove the linear map $T$ extends to a bounded map, it suffices to prove that it is bounded on the dense $C^\infty$. We have, for any $t \in [0, 1]$,
\[
T(f)(t)^2 = \left( \int_0^t f(s)ds \right)^2 \leq t \left( \int_0^t f(s)^2ds \right) \leq \int_0^t f(s)^2ds
\]
and therefore also
\[
\int_0^1 T(f)(t)^2dt \leq \int_0^t f(s)^2ds.
\]
Thus we have $||T(f)||^2_H \leq 2||f||^2_{L^2} \leq 2||f||^2_H$.

If one consider the constant function $f \equiv 1$, then $||f||_H = 1$ but $||T(f)||_H > 1$. This shows the norm must be greater than 1.

(b) The plan is to apply the Arzela-Ascoli theorem. For a bounded sequence $f_1, ..., f_n, ...$ in $\mathbb{H}$, as the operator is bounded by further approximation we are free to assume each $f_i \in C^\infty$ and we have to prove $\{T(f_i)\}$ has a convergent subsequence. We have, for any $t_1, t_2 \in [0, 1],
\[
|f_i(t_1) - f_i(t_2)| = \left| \int_{t_1}^{t_2} f_i'(s) ds \right| \leq \left| t_1 - t_2 \right| \left( \int_0^1 f_i'(s)^2 ds \right)^{1/2}
\]
is bounded. Also
\[
\inf_{t \in [0,1]} f_i(t) \leq \left( \int_0^1 f_i(s)^2 ds \right)^{1/2}.
\]
These together show that $f_i$ are uniformly bounded and equicontinuous. Thus by the Arzela-Ascoli theorem these $f_i$ have a uniformly convergent subsequence, thus a $L^2$ convergent subsequence. As we’ve seen in (a) that the $\mathbb{H}$-norm of $T(f)$ is bounded by the $L^2$-norm of $f$, this gives us a convergent subsequence $T(f_i)$ in $\mathbb{H}$.

(c) The second last inequality just used is just
\[
\frac{|f(t_1) - f(t_2)|}{|t_1 - t_2|^{1/2}} \leq \left( \int_0^1 f'(s)^2 ds \right)^{1/2}.
\]

Also by the two inequalities in (b) sup $f$ is bounded when $f$ has bounded $\mathbb{H}$-norm. Thus the map from $C^\infty$ to $C^{1/2}$ is bounded with respect to the $\mathbb{H}$-norm on $C^\infty$, and therefore extends.

(d) Let $g_0 : [0, +\infty) \to \mathbb{R}$ be any nonzero smooth function supported only on $[\frac{1}{2}, 1]$. Let $g_{n+1}(t) = \frac{1}{2}g_n(4t)$ for any $n \geq 0$. Then these $g_i$ have disjoint support. Note that $||g_{n+1}||_{L^2} = \frac{1}{2}||g_n||_{L^2}$ and $||g_{n+1}'||_{L^2} = ||g_n'||_{L^2}$. Thus $||g_n||_{\mathbb{H}}$ converges to $||g_0||_{L^2} \neq 0$. Similarly,
\[
\sup_{t_1 \neq t_2} \frac{|g_{n+1}(t_1) - g_{n+1}(t_2)|}{|t_1 - t_2|^{1/2}} = \sup_{t_1 \neq t_2} \frac{|g_n(t_1) - g_n(t_2)|}{|t_1 - t_2|^{1/2}} \neq 0
\]
and \( \sup g_{n+1} = \frac{1}{2} \sup g_n \). Thus \( ||g_n||_{C^{1/2}} \) also converges to some positive number (which is finite by (c)).

We can now normalize each \( g_n \) so that \( ||g_n||_\infty = 1 \), and still have \( ||g_n||_{C^{1/2}} \) converges to a positive number. As these \( g_n \) have disjoint support, \( ||g_n - g_m||_{C^{1/2}} \geq \max(||g_n||_{C^{1/2}}, ||g_m||_{C^{1/2}}) \) and thus they have no convergent subsequence.