QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday August 31, 2010 (Day 1)

1. (CA) Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx$$

Solution. Let C be the curve on the complex plane from $-\infty$ to $+\infty$, which is along the real line for most part but gets around the origin by going upwards (clockwise). We are integrating

$$\int_C \frac{\sin^2 z}{z^2} dz = \int_C \frac{2 - e^{2iz} - e^{-2iz}}{4z^2} dz = \int_C \frac{1 - e^{2iz}}{4z^2} dz + \int_C \frac{1 - e^{-2iz}}{4z^2} dz.$$

Let C' be the curve from $-\infty$ to $+\infty$, along the real line for most part but now goes downwards around the origin. Then

$$\int_C \frac{1 - e^{-2iz}}{4z^2} dz - \int_{C'} \frac{1 - e^{-2iz}}{4z^2} dz = -2\pi i \cdot \mathrm{Res}_{z=0} \left(\frac{1 - e^{-2iz}}{4z^2} \right) = \pi.$$

As $1 - e^{2iz}$ is bounded when $\mathrm{Im}(z) \geq 0$, $\int_C \frac{1-e^{2iz}}{4z^2} dz = 0$ as we can push the integral up to infinity. Similarly $\int_{C'} \frac{1-e^{-2iz}}{4z^2} dz = 0$. This shows the original integral has value π.

2. (A) Let b be any integer with $(7, b) = 1$ and consider the polynomial

$$f_b(x) = x^3 - 21x + 35b.$$

(a) Show that f_b is irreducible over $\mathbb{Q}$.

(b) Let P denote the set of $b \in \mathbb{Z}$ such that $(7, b) = 1$ and the Galois group of f_b is the alternating group A_3. Find P.

Solution.

(a) This follows from the Eisenstein criterion on the prime 7.

(b) From (a), the Galois group is A_3 if the discriminant is a square (in $\mathbb{Q}$), and S_3 if otherwise. The discriminant of $x^3 + ax + b$ is $-4a^3 - 27b^2$, and the discriminant of f_b is $4 \cdot 21^3 - 27 \cdot 35^2 \cdot b^2 = 3^3 7^2 (28 - 25b)$. Thus we're looking for all b such that $3(28 - 25b)$ is a square. Such square must be divisible by 9 and are congruent to 9 modulo 25, hence of the form $(75n \pm 3)^2$, i.e. $3(28 - 25b) = 5625n^2 \pm 450n + 9 \Leftrightarrow b = -75n^2 \pm 6n + 1$. Thus $P = \{-75n^2 + 6n + 1 \,|\, n \in \mathbb{Z}\}$.

3. (T) Let X be the Klein bottle, obtained from the square $I^2 = \{(x, y) : 0 \leq x, y \leq 1\} \subset \mathbb{R}^2$ by the equivalence relation $(0, y) \sim (1, y)$ and $(x, 0) \sim (1-x, 1)$.

(a) Compute the homology groups $H_n(X, \mathbb{Z})$.

(b) Compute the homology groups $H_n(X, \mathbb{Z}/2)$.

(c) Compute the homology groups $H_n(X \times X, \mathbb{Z}/2)$.

Solution. X has the following cellular decomposition: the square F, the edges $E_1 = \{0\} \times [0, 1]$ and $E_2 = [0, 1] \times \{0\}$, and the vertex $V = (0, 0)$. We have $\delta F = 2E_1$ and $\delta E_1 = \delta E_2 = \delta V = 0$.

(a) $H_2(X, \mathbb{Z}) = \{c \cdot F | \delta(c \cdot F) = 0\} = 0$. As all other boundary maps are zero, $H_1(X, \mathbb{Z}) = (\mathbb{Z}E_1 + \mathbb{Z}E_2)/2\mathbb{Z}E_1 \cong (\mathbb{Z}/2) \oplus \mathbb{Z}$ and $H_0(X, \mathbb{Z}) = \mathbb{Z}$.

(b) All boundary maps are zero in $\mathbb{Z}/2$-coefficient. Thus $H_2(X, \mathbb{Z}/2) = \mathbb{Z}/2$, $H_1(X, \mathbb{Z}/2) = (\mathbb{Z}/2)^2$ and $H_0(X, \mathbb{Z}/2) = \mathbb{Z}/2$.

(c) $\mathbb{Z}/2$ may be seen as a field. Thus $H^i(X, \mathbb{Z}/2) = H_i(X, \mathbb{Z}/2)$ for any i and by the Kunneth formula $H_*(X \times X, \mathbb{Z}/2) = H^*(X \times X, \mathbb{Z}/2) = H^*(X, \mathbb{Z}/2)^{\otimes 2}$. Explicitly

$$H^i(X \times X, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & i = 0, 4 \\ (\mathbb{Z}/2)^4 & i = 1, 3 \\ (\mathbb{Z}/2)^6 & i = 2 \\ 0 & \text{else} \end{cases}$$

4. (RA) Let f be a Lebesgue integrable function on the closed interval $[0, 1] \subset \mathbb{R}$.

(a) Suppose g is a continuous function on $[0, 1]$ such that the integral of $|f - g|$ is less than ϵ^2. Prove that the set where $|f - g| > \epsilon$ has measure less than ϵ.

(b) Show that for every $\epsilon > 0$, there is a continuous function g on $[0, 1]$ such that the integral of $|f - g|$ is less than ϵ^2.

Solution.

(a) This is obvious.

(b) We have to prove that continuous functions are dense As f is Lebesgue integrable, f can be L^1-approximated by step functions, i.e. for any $\delta > 0$, there exist real numbers $c_1, ..., c_n$ and measurable sets $E_1, ..., E_n \subset [0, 1]$ such that the integral of $|f - c_1\chi_{E_1} - ... - c_n\chi_{E_n}|$ is smaller than δ, where we denote by χ_E the characteristic function of E. By picking small enough δ and replace f by $c_1\chi_{E_1} + ... + c_n\chi_{E_n}$, it suffices to prove that for any $\epsilon > 0$ and any characteristic function χ_E of a measurable set $E \subset [0, 1]$, there is a continuous function g_E such that the integral of $|g_E - \chi_E|$ is smaller than ϵ

As the Lebesgue measure is inner and outer regular, we may find compact K and open U such that $K \subset E \subset U$ and the measure of $U - K$ is arbitrarily small. Urysohn lemma now gives us a continuous function that is 1 on K, 0 on $[0,1] - U$ and between 0 and 1 in $U - K$. This gives the required function g_E.

5. (DG) Let v denote a vector field on a smooth manifold M and let $p \in M$ be a point. An *integral curve* of v through p is a smooth map $\gamma : U \to M$ from a neighborhood U of $0 \in \mathbb{R}$ to M such that $\gamma(0) = p$ and the differential $d\gamma$ carries the tangent vector $\partial/\partial t$ to $v(\gamma(t))$ for all $t \in U$.

(a) Prove that for any $p \in M$ there is an integral curve of v through p.

(b) Prove that any two integral curves of v through any given point p agree on some neighborhood of $0 \in \mathbb{R}$.

(c) A *complete* integral curve of v through p is one whose associated map has domain the whole of $\mathbb{R}$. Give an example of a nowhere zero vector field on $\mathbb{R}^2$ that has a complete integral curve through any given point. Then, give an example of a nowhere zero vector field on $\mathbb{R}^2$ and a point which has no complete integral curve through it.

Solution. Pick a local chart of the manifold M at the considered point p. The chart may be seen as a neighborhood of a point $p \in \mathbb{R}^n$, and the vector field v is also given on the neighborhood. To give an integral curve through a point p is then to solve the ordinary differential equation (system) $x'(t) = v(x(t))$ and $x(0) = p$. As v is smooth and thus C^1, (a) and (b) follows from the (local) existence and uniqueness of solutions for ordinary differential equations.

For (c), constant vector field $v(x, y) = (1, 0)$ on $\mathbb{R}^2$ gives the first required example. Horizontal curves parametrized by arc length are all possible integral curves. For the second required example, we may consider $v(x, y) = (x^2, 0)$. A integral curve with respect to such a vector field is a solution to the ODE $x'(t) = x(t)^2$. Such a solution is of the form $\frac{1}{t-a}$, and always blows up in finite time (either forward or backward), i.e. there is no complete integral curve for this vector field.

6. (AG) Show that a general hypersurface $X \subset \mathbb{P}^n$ of degree $d > 2n - 3$ contains no lines $L \subset \mathbb{P}^n$.

Solution. A hypersurface in $\mathbb{P}^n$ of degree d is given by a homogeneous polynomial in $n + 1$ variable $x_0, x_1, ..., x_n$ of degree d up to a constant. There are $k(d, n) = \binom{d+n}{n}$ such monomials, and thus the space of such polynomials is $\mathbb{P}^{k(d,n)-1}$. After a change of coordinate a line may be expressed as $x_2 = x_3 = ... = x_n = 0$. A hypersurface that contains this line then corresponds to a polynomial with no $x_0^d, x_0^{d-1}x_1, ..., x_1^d$ terms, which constitutes a codimension $d + 1$ subplane. On the other hand, the grassmannian of lines is a variety of dimension $2 \cdot ((n+1)-2) = 2n-2 < d+1$. This proves the assertion.

To be more rigerous, let G be the grassmannian of lines in $\mathbb{P}^n$, $H \cong \mathbb{P}^{k(d,n)-1}$ the space of hypersurfaces of degree d in $\mathbb{P}^n$. We may consider

$$X = \{(l, S) \mid l \in G,\ S \in H \text{ such that } l \subset S\}.$$

Then what we have learned is that G has dimension $2n-2$ and the fiber of the projection map $X \to G$ has dimension $d+1$ less than the dimension of H. Thus the dimension of X is the sum of the dimension of G and the dimension of the fiber H, which is smaller than the dimension of H exactly when $d > 2n-3$. It follows that the projection $X \to H$ cannot be surjective, which is the assertion to be proved.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday September 1, 2010 (Day 2)

1. (T) If M_g denotes the closed orientable surface of genus g, show that continuous maps $M_g \to M_h$ of degree 1 exist if and only if $g \geq h$.

Solution. A closed orientable surface of genus $g \geq 1$ may be described as a polygon of $4g$ edges, some pairs identified in a certain way. In particular, all vertices are identified together under this identification. For $g > h$, by further identify $4g - 4h$ edges to the point, we can construct a map from M_g to M_h which is a homeomorphism on the interior of the polygon (2-cell). Since the 2-cell is the generator of $H_2(\cdot, \mathbb{Z})$, the map constructed has degree 1.

If $g < h$, then any map $f : M_g \to M_h$ induces $f^* : H^1(M_h, \mathbb{Z}) \to H^1(M_g, \mathbb{Z})$, which cannot be injective since the former is a free abelian group of rank $2h$ and the latter has rank $2g$. Pick $0 \neq \alpha \in H^1(M_h, \mathbb{Z})$ such that $f^*(\alpha) = 0$, there always exists β with $\alpha.\beta \neq 0 \in H^2(M_h, \mathbb{Z})$. However $f^*(\alpha.\beta) = f^*(\alpha).f^*(\beta) = 0$, and thus f must has degree 0.

2. (RA) Let $f \in C(S^1)$ be continuous function with a continuous first derivative $f'(x)$. Let $\{a_n\}$ be the Fourier coefficient of f. Prove that $\sum_n |a_n| < \infty$.

Solution. f' has n-th Fourier coefficient equal to na_n. We thus have

$$||f'||_{L^2}^2 = \sum_n n^2 a_n^2 < \infty.$$

Then $(\sum_n |a_n|)^2 \leq (\sum_n n^2 a_n^2)(\sum_n 1/n^2) < \infty$ by Cauchy's inequality.

3. (DG) Let $S \subset \mathbb{R}^3$ be the surface given as aa graph

$$z = ax^2 + 2bxy + cy^2$$

where a, b and c are constants.

(a) Give a formula for the curvature at $(x, y, z) = (0, 0, 0)$ of the induced Riemannian metric on S.

(b) Give a formula for the second fundamental form at $(x, y, z) = (0, 0, 0)$.

(c) Give necessary and sufficient conditions on the constants a, b and c that any curve in S whose image under projection to the (x, y)-plane is a straight line through $(0, 0)$ is a geodesic on S.

Solution. Let normal vectors to the surface may be expressed as $n(x, y, z) = l(x, y, z) \cdot (ax + by, bx + cy, -1)$, where $l(x, y, z)$ is the inverse of the length of the vector. Note that $l(x,y,z) = 0$ and the first derivative of $l(x,y,z)$ is zero at $(0,0,0) \in S$. When we compute the second fundamental form we only have to compute the first derivative of $n(x,y,z)$. Therefore to compute the second fundamental form at $(0,0,0)$ we can treat $l(x,y,z) \equiv 1$ and the second fundamental form is thus

$$\begin{pmatrix} \frac{\partial}{\partial x}(ax+by) & \frac{\partial}{\partial y}(ax+by) \\ \frac{\partial}{\partial x}(bx+cy) & \frac{\partial}{\partial y}(bx+cy) \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

The curvature of the surface at $(0,0,0)$ is the determinant at the point, i.e. $ac - b^2$. For any curve whose projection to the (x,y)-plane is a straight line through the point to be a geodesic, the corresponding vector has to be an eigenvector of the matrix. Thus it is necessary that $a = c$, $b = 0$, i.e. the matrix being a multiple of the identity matrix for that to happen. On the other hand, when $a = c$, $b = 0$, the surface is radially symmetric and thus all such curves must be geodesics.

4. (AG) Let V and W be complex vector spaces of dimensions m and n respectively and $A \subset V$ a subspace of dimension l. Let $\mathbb{P}\mathrm{Hom}(V,W) \cong \mathbb{P}^{mn-1}$ be the projective space of nonzero linear maps $\phi : V \to W$ mod scalars, and for any integer $k \le l$ let

$$\Psi_k = \{\phi : V \to W \ : \ \mathrm{rank}(\phi|_A) \le k\} \subset \mathbb{P}^{mn-1}.$$

Show that Ψ_k is an irreducible subvariety of $\mathbb{P}^{mn-1}$, and find its dimension.

Solution. An $n \times m$ matrix of rank $\le k$ can be decomposed into the product of an $n \times k$ matrix and a $k \times m$ matrix. Let $X \cong \mathbb{P}^{nk-1}$ and $Y \cong \mathbb{P}^{km-1}$ be the space of nonzero such matrices mod scalars. Then we have a surjection $X \times Y \to \Psi_k$ by the multiplication map. This shows that Ψ_k, as the image of the complete irreducible variety $X \times Y$, is irreducible and closed in $\mathbb{P}^{mn-1}$.

When a matrix has rank exactly k, the decomposition has a $GL(k)$ freedom of choice, i.e. each fiber of this map over a point in $\Psi_k - \Psi_{k-1}$ has dimension $k^2 - 1$. As Ψ_{k-1} is closed in irreducible Ψ_k, $\dim \Psi_k = \dim(\Psi_k - \Psi_{k-1}) = \dim(X \times Y) - (k^2 - 1) = (nk-1) + (mk-1) - (k^2-1) = k(n+m-k) - 1$. (We used the fact that $k \le l$, in which case $\Psi_k - \Psi_{k-1}$ is obviously non-empty.)

5. (CA) Find a conformal map from the region

$$\Omega = \{z : |z-1| > 1 \text{ and } |z-2| < 2\} \subset \mathbb{C}$$

between the two circles $|z-1| = 1$ and $|z-2| = 2$ onto the upper-half plane.

Solution. Let $S = \{\frac{1}{4} \le \mathrm{Re}(z) \le \frac{1}{2}\} \subset \mathbb{C}$. Then we have $\Omega \cong S$ by $z \mapsto \frac{1}{z}$ and $S \cong$ upper-half plane by $z \mapsto e^{2\pi i (z - \frac{1}{4})}$.

6. (A) Let G be a finite group with an automorphism $\sigma : G \to G$. If $\sigma^2 = id$ and the only element fixed by σ is the identity of G, show that G is abelian.

Solution. Define $\tau(x) := \sigma(x)x^{-1}$, then by assumption $\tau(x) \neq e,\ \forall x \neq e$. For any $x \neq x'$, $\tau(x)\tau(x')^{-1} = \sigma(x)x^{-1}x'\sigma(x')^{-1}$ is conjugate to $\sigma(x')^{-1}\sigma(x)x^{-1}x' = \tau(x'^{-1}x) \neq e$, i.e. $\tau(x) \neq \tau(x')$. Thus $\tau : G \to G$ is a surjective function. But we have $\sigma(\tau(x)) = x\sigma(x)^{-1} = \tau(x)^{-1}$, hence $\sigma(x) = x^{-1}$ and G is abelian.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Thursday September 2, 2010 (Day 3)

1. (DG) Let $D \subset \mathbb{R}^2$ be the closed unit disk, with boundary $\partial D \cong S^1$. For any smooth map $\gamma : D \to \mathbb{R}^2$, let $A(\gamma)$ denote the integral over D of the pull-back $\gamma^*(dx \wedge dy)$ of the area 2-form $dx \wedge dy$ on $\mathbb{R}^2$.

(a) Prove that $A(\gamma) = A(\gamma')$ if $\gamma = \gamma'$ on the boundary of D.

(b) Let $\alpha : \partial D \to \mathbb{R}^2$ denote a smooth map, and let $\gamma : D \to \mathbb{R}^2$ denote a smooth map such that $\gamma|_{\partial D} = \alpha$. Give an expression for $A(\gamma)$ as an integral over ∂D of a function that is expressed only in terms of α and its derivatives to various orders.

(c) Give an example of a map γ such that $\gamma^*(dx \wedge dy)$ is a positive multiple of $dx \wedge dy$ at some points and a negative multiple at others.

Solution. Consider the differential $\omega = ydx$ on $\mathbb{R}^2$, so $d\omega = dx \wedge dy$. We have $\gamma^*(d\omega) = d\gamma^*(\omega)$. Thus if $\gamma|_{\partial D} = \alpha$, the by Stoke's theorem

$$\int_D \gamma^*(dx \wedge dy) = \int_D d\gamma^*(\omega) = \int_{\partial D} \alpha^*(\omega).$$

and depends only on α instead of γ. This finishes both (a) and (b).

For (c), one take for example $\gamma(x, y) = (x^2, y)$, then $\gamma^*(dx \wedge dy) = 2x(dx \wedge dy)$.

2. (T) Compute the fundamental group of the space X obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

Solution. The space X is $S^1 \times Y$, where Y is obtained from two circle S^1 by identifying a point $x_0 \in S^1$ with the corresponding point on the other circle. Thus $\pi_1(X) = \pi_1(S^1) \times \pi_1(Y)$, the product of $\mathbb{Z}$ and a free group on two generators.

3. (CA) Let u be a positive harmonic function on $\mathbb{C}$. Show that u is a constant.

Solution. There exists a holomorphic function f on $\mathbb{C}$ such that u is the real part of it. Then e^{-f} has image in the unit disk. By Liouville's theorem e^{-f} must be a constant, hence so is u.

4. (A) Let $R = \mathbb{Z}[\sqrt{-5}]$. Express the ideal $(6) = 6R \subset R$ as a product of prime ideals in R.

Solution. $(6) = (2)(3)$ and $(2) = (2, 1 + \sqrt{-5})^2$, $(3) = (3, 1 + \sqrt{-5})(3, 1 - \sqrt{-5})$. The final resulting ideals are prime because their indices (to R) are prime numbers.

5. (AG) Let $Q \subset \mathbb{P}^5$ be a smooth quadric hypersurface, and $L \subset Q$ a line. Show that there are exactly two 2-planes $\Lambda \cong \mathbb{P}^2 \subset \mathbb{P}^5$ contained in Q and containing L.

Solution. By a linear change of coordinate we may assume the line is $x_2 = x_3 = x_4 = x_5 = 0$, where $x_0, ..., x_5$ are coordinates of the projective space $\mathbb{P}^5$. Then the degree two homogeneous polynomial defining Q may be written as $F = f_0(x_2, x_3, x_4, x_5)x_0 + f_1(x_2, x_3, x_4, x_5)x_1 + q(x_2, x_3, x_4, x_5)$, where neither f_0 nor f_1 are a constant multiple of the other since $\frac{\partial F}{\partial x_0}$ and $\frac{\partial F}{\partial x_1}$ have to be independent for Q to be smooth. We may thus arrange another change of coordinate among x_2, x_3, x_4, x_5 so that $f_0 = x_2$, $f_1 = x_3$. The $F = x_0x_2 + x_1x_3 + q(x_2, x_3, x_4, x_5)$.

Any plane that lies within Q and contains L is then of the form $(x_0 = x_1 = 0, ax_4 + bx_5 = 0)$, where $ax_4 + bx_5$ is nontrivial and divides $q(0, 0, x_4, x_5)$. Note $\frac{\partial F}{\partial x_0} = x_2$ and $\frac{\partial F}{\partial x_1} = x_3$. For Q to be smooth we need $\frac{\partial F}{\partial x_i}$ to be independent, thus $\frac{\partial F}{\partial x_4}q(0, 0, x_4, x_5)$ and $\frac{\partial F}{\partial x_5}q(0, 0, x_4, x_5)$ have to be independent, which is equivalent to $q(0, 0, x_4, x_5)$ is non-degenerate, in which case it has two linear divisors.

6. (RA) Let $\mathcal{C}^\infty$ denote the space of smooth, real-valued functions on the closed interval $I = [0, 1]$. Let $\mathbb{H}$ denote the completion of $\mathcal{C}^\infty$ using the norm whose square is the functional

$$f \mapsto \int_I \left((\frac{df}{dt})^2 + f^2 \right) dt.$$

(a) Prove that the map of $\mathcal{C}^\infty$ to itself given by $f \mapsto T(f)$ with

$$T(f)(t) = \int_0^t f(s)ds$$

extends to give a bounded map from $\mathbb{H}$ to $\mathbb{H}$, and prove that the norm of T is 1. *(Remark: Its norm is actually not* 1*)*

(b) Prove that T is a compact mapping from $\mathbb{H}$ to $\mathbb{H}$.

(c) Let $\mathcal{C}^{1/2}$ be the Banach space obtained by completing $\mathcal{C}^\infty$ using the norm given by

$$f \mapsto \sup_{t \neq t'} \frac{|f(t) - f(t')|}{|t - t'|^{1/2}} + \sup_t |f(t)|.$$

Prove that the inclusion of $\mathcal{C}^\infty$ into $\mathbb{H}$ and into $\mathcal{C}^{1/2}$ extends to give a bounded, linear map from $\mathbb{H}$ to $\mathcal{C}^{1/2}$.

(d) Give an example of a sequence in $\mathbb{H}$ such that all elements have norm 1 and such that there are no convergent subsequences in $\mathcal{C}^{1/2}$.

Solution.

(a) To prove the linear map T extends to a bounded map, it suffices to prove that it is bounded on the dense $\mathcal{C}^\infty$. We have, for any $t \in [0,1]$,

$$T(f)(t)^2 = \left(\int_0^t f(s)ds\right)^2 \leq t\left(\int_0^t f(s)^2 ds\right) \leq \int_0^t f(s)^2 ds$$

and therefore also

$$\int_0^1 T(f)(t)^2 dt \leq \int_0^1 f(s)^2 ds.$$

Thus we have $||T(f)||_{\mathbb{H}}^2 \leq 2||f||_{L^2}^2 \leq 2||f||_{\mathbb{H}}^2$.
If one consider the constant function $f \equiv 1$, then $||f||_{\mathbb{H}} = 1$ but $||T(f)||_{\mathbb{H}} > 1$. This shows the norm must be greater than 1.

(b) The plan is to apply the Arzela-Ascoli theorem. For a bounded sequence $f_1, ..., f_n, ...$ in $\mathbb{H}$, as the operator is bounded by further approximation we are free to assume each $f_i \in \mathcal{C}^\infty$ and we have to prove $\{T(f_i)\}$ has a convergent subsequence. We have, for any $t_1, t_2 \in [0,1]$,

$$|f_i(t_1) - f_i(t_2)| = |\int_{t_1}^{t_2} f_i'(s)ds| \leq \left(|t_1 - t_2| \int_0^1 f_i'(s)^2 ds\right)^{1/2}$$

is bounded. Also

$$\inf_{t\in[0,1]} f_i(t) \leq \left(\int_0^1 f_i(s)^2 ds\right)^{1/2}.$$

These toghther show that f_i are uniformly bounded and equicontinuous. Thus by the Arzela-Ascoli theorem these f_i have a uniformly convergent subsequence, thus a L^2 convergent subsequence. As we've seen in (a) that the $\mathbb{H}$-norm of $T(f)$ is bounded by the L^2-norm of f, this gives us a convergent subsequence $T(f_i)$ in $\mathbb{H}$.

(c) The second last inequality just used is just

$$\frac{|f(t_1) - f(t_2)|}{|t_1 - t_2|^{1/2}} \leq \left(\int_0^1 f'(s)^2 ds\right)^{1/2}.$$

Also by the two inequalities in (b) $\sup f$ is bounded when f has bounded $\mathbb{H}$-norm. Thus the map from $\mathcal{C}^\infty$ to $\mathcal{C}^{1/2}$ is bounded with respect to the $\mathbb{H}$-norm on $\mathcal{C}^\infty$, and therefore extends.

(d) Let $g_0 : [0, +\infty) \to \mathbb{R}$ be any nonzero smooth function supported only on $[\frac{1}{2}, 1]$. Let $g_{n+1}(t) = \frac{1}{2}g_n(4t)$ for any $n \geq 0$. Then these g_i have disjoint support. Note that $||g_{n+1}||_{L^2} = \frac{1}{4}||g_n||_{L^2}$ and $||g_{n+1}'||_{L^2} = ||g_n'||_{L^2}$. Thus $||g_n||_{\mathbb{H}}$ converges to $||g_0'||_{L^2} \neq 0$. Similarly,

$$\sup_{t_1 \neq t_2} \frac{|g_{n+1}(t_1) - g_{n+1}(t_2)|}{|t_1 - t_2|^{1/2}} = \sup_{t_1 \neq t_2} \frac{|g_n(t_1) - g_n(t_2)|}{|t_1 - t_2|^{1/2}} \neq 0$$

and $\sup g_{n+1} = \frac{1}{2} \sup g_n$. Thus $||g_n||_{\mathcal{C}^{1/2}}$ also converges to some positive number (which is finite by (c)).

We can now normalize each g_n so that $||g_n||_{\mathbb{H}} = 1$, and still have $||g_n||_{\mathcal{C}^{1/2}}$ converges to a positive number. As these g_n have disjoint support, $||g_n - g_m||_{\mathcal{C}^{1/2}} \geq \max(||g_n||_{\mathcal{C}^{1/2}}, ||g_m||_{\mathcal{C}^{1/2}})$ and thus they have no convergent subsequence.