The theory of group representations is in the center of interest of I. Gelfand. I think that this is related to the nature of this domain which combines analysis, algebra and topology in a very intricate fashion. But this richness of the Representation Theory should not be taken as self evident. To a great extent we owe this understanding to works of I. Gelfand, to his unique way to see mathematics as a unity of different points of view.

In the late thirties, when Gelfand started his mathematical career, the theory of representations of compact groups and the general principles of harmonic analysis on compact groups was well understood due to works of Herman Weyl. Harmonic analysis on locally compact abelian groups was developed in works of Pontryagin. The general structure of operator algebras was clarified in works of Murrey-Von Neumann. But the representation theory of non-compact non-commutative groups was almost non-existent. The only result I know is the work of E. Wigner on representations of the inhomogeneous Lorentz group. Wigner has shown that the study of physically interesting irreducible representations of this group can be reduced to the study of irreducible representations of it's compact subgroups.

It was not at all clear whether the theory of representations of real semisimple non-compact groups is "good", i.e., whether the set of irreducible representations could be parameterized by points of a "reasonable" set, and whether the unitary representations can be uniquely decomposed into irreducible ones. The "conventional wisdom" was to expect that the beautiful theory of Murrey-Von Neumann factors is necessary for the description of representations of real semisimple non-compact groups. On the other hand, Gelfand, for whom Gauss and Riemann are the heroes, expected that the theory of representations of such groups should possess the classical beauty.

Gelfand's first result [1942] (with Rajkov) in the theory of representations of groups is a proof of the existence of "sufficiently many" unitary irreducible representations for any locally compact group G. In other words, any unitary representation of G is a direct integral of irreducible ones. The proof of this result is based on the very important observation that the representation theory of the group G is identical to the theory of representations of the convolution algebra $\mathbb{C}_c(G)$ of measures on G with compact support, and on an application of Gelfand's theory of normed rings.

Next, in the late forties there was a stream of papers (most of them joint with Naimark) which developed the main concepts of the representation theory of complex classical groups G. It would be a much simpler task to describe the concepts which appeared later than to describe the richness of these works.

Gelfand believed that the space $\hat{G}$ of irreducible representations of G is a reasonable "classical" space. If I understand correctly, the first indication of the correctness of this intuition came from the theory of spherical functions, developed in early forties but published only in 1950. Let $K \subset G$ be the maximal compact subgroup and $\hat{G}_0 \subset \hat{G}$ be the subset of irreducible representations of *class 1* [that is, such representations (π, V) of G that $V^K \neq 0$]. Gelfand observed that the subset $\hat{G}_0$ is equal to the set of irreducible representations of the subalgebra $\mathbb{C}_c(K\backslash G/K) \subset \mathbb{C}_c(G)$ of two-sided K-invariant functions on G, proved the commutativity of this algebra, and identified the space of maximal ideals of $\mathbb{C}_c(K\backslash G/K)$ with the quotient ${}^LT/W$ where ${}^LT$ is the torus dual to the maximal split torus

$T \subset G$ and W is the Weyl group. The generalization of this approach developed in 50-s by Harish-Chandra and Godement led to the proof of the uniqueness of the decomposition of any representation of the group G into irreducibles.

Given such a nice classification of irreducible representations of class 1, it was natural to guess that the total space $\hat{G}$ is also an algebraic variety.

But for this purpose one had to find a way to construct irreducible representations of G. Gelfand introduced the notion of parabolic induction [for classical groups] and in particular studied representations π_χ of G induced from a character χ of a Borel subgroup $B \subset G$. He showed that for generic character χ of $T \equiv B/U$ the representation π_χ is irreducible, and the representations π_χ, π'_χ are equivalent if and only if the characters χ, χ' of T are conjugate under the action of the Weyl group W. The proof is based on the the decomposition $G = \cup_{w \in W} BwB$ for classical groups. This decomposition was extended by Harish-Chandra to the case of an arbitrary semisimple group and is known now as the Bruhat decomposition.

This construction gives many irreducible representations. But how to show that not much is missing? In the case of a compact group G it is well known that all the representations of G are constituents of the regular representation. Therefore, to see that a list of representations $\pi_a, a \in A$ of G is complete, it is sufficient to show that one can write the delta function δ_e on G as a linear combination of the characters $tr(\pi_a)$. But for representations (π, V) of a non-compact group G, which are typically infinite dimensional, the trace of the operator $\pi(g), g \in G$ is not defined. The ingenious idea of Gelfand was to define characters $tr(\pi)$ as *distributions*. That is, he showed that for any smooth function $f(g)$ with compact support, the operator $\pi_\chi(f) := \int f(g)\pi_\chi(g)dg$ is of trace class, and defined the distribution $tr(\pi)$ by $tr(\pi)(f) := tr\pi(f)$. Now one could look for a W-invariant measure [called the *Plancherel measure*] μ_X on the space X of unitary characters of T such that

$$\delta_e = \int_{\chi \in X} tr(\pi_\chi)\mu_X \tag{1}$$

It is not difficult to see that such a measure μ_X is unique [if it exists] and the knowledge of μ_X is equivalent to the explicit decomposition of the regular representation $L^2(G)$ into the irreducible ones.

Gelfand [in a series of joint works with Naimark] was able to guess a beautiful algebraic expression for the Plancherel measure μ_X in the case of classical complex groups, and to prove equality (1) by a very intricate explicit calculation - a great reward for difficult work.

As a continuation of this series of works, Gelfand asked a number of questions [which he was able to answer only in particular cases], which influenced the development of representation theory for many years.

a) Gelfand [in a joint work with Graev] classified generic irreducible representations of the group $SL(n, \mathbb{R})$. They found that $\hat{G}$ is a union of pieces called *series* which correspond to conjugacy classes of maximal tori. Moreover the series corresponding to non-split tori have realizations in spaces of [partially] analytic functions. Gelfand conjectured that the analogous description of the space $\hat{G}$ should be true for all real semisimple groups and it should be possible to realize discrete series in appropriate spaces of analytic functions. The first part of the conjecture was justified by Harish-Chandra [who constructed discrete series of representations for real semisimple groups G, and found the Plancherel measure concentrated on representations unitary induced from discrete series of Levi subgroups]. The second part was modified by Langlands who suggested the realization of discrete series [which exist iff there is a compact maximal torus$T^c \subset G$] in the space of cohomologies $H^i(G/T^c, \mathcal{F})$ of homogeneous holomorphic vector bundles $\mathcal{F}$ on G/T^c [while Gelfand considered only the realization in sections of such bundles].

b) Gelfand [in a joint work with Graev] constructed the analog of the Paley-Wiener theorem for groups $SL_2(\mathbb{C})$ and $SL_2(\mathbb{R})$ [that is, a decomposition of the representation of G on the space $\mathbb{C}_c^\infty(G)$ of smooth functions with compact support], and raised the question about the extension of this result to other groups. This generalization was obtained by J. Arthur in 1983.

c) Gelfand [also in a joint work with Graev] constructed the decomposition of representations of the group $SL_2(\mathbb{C})$ on the space $L^2(SL_2(\mathbb{C})/SL_2(\mathbb{R})$, and asked the question about the decomposition of representations of the group G on $L^2(G/H)$ where $H \subset G$ is the fixed points of an involution. An extension of this result to arbitrary such pair (G, H) is achieved very recently [see the talk of P. Delorme at the ICM congress of 2002].

d) Gelfand has shown that many special functions such as Bessel and Whittaker functions, Jacobi and Legendre polynomials appear as matrix coefficients of irreducible representations. This interpretation of special functions immediately explains the functional and differential equations for these functions. It is clear now that [almost] all special functions studied in the 19-20 centuries can be interpreted as matrix coefficients or traces of representations of groups or their quantum analogs (e.g., works of Tsuchiya-Kanie, I. Frenkel-Reshetikhin on the representation theoretic interpretation of hypergeometric (respectively q-hypergeometric) functions, works of Koornwinder, Koelink, Noumi, Rosengren, Stokman, Sugitani and others on representation theoretic interpretation of Askey-Wilson, Macdonald, and Koornwinder polynomials).

The next series of Gelfand's works [with Tsetlin] is on irreducible finite-dimensional representations of classical groups G. The classification of such representations (π_λ, V_λ) was well known but Gelfand asked a new question, partially influenced by his interest in physics- how to find a "good" realization of these representations. In other words, how to find a basis in V_λ such that it is possible to compute matrix coefficients of $\pi_\lambda(g), g \in G$ in these bases. Such a basis [called the Gelfand-Tsetlin basis] was constructed for irreducible representations of groups SL_n and SO_n, and is now a core of many works in representation theory and combinatorics.

Gelfand and Graev found the expression for the matrix coefficients of the representations π_λ in the terms of discrete versions of Γ-functions. This realization of finite-dimensional representations has an important analog for infinite-dimensional representations of groups over local fields.

As a part of the theory of finite-dimensional representations Gelfand studied the Clebsch-Gordan coefficients which give a decomposition of the tensor products of irreducible representations into irreducible components. He noticed that [at least in the case $G = SL_2$] the Clebsch-Gordon coefficients of G are discrete analogs of the Jacobi polynomials which are matrix coefficients of irreducible representations of G. Possibly an explanation of this can be given using the theory of quantum groups where multiplication and comultiplication are almost symmetric to each other.

The next series of Gelfand's works is on representations of groups over finite and local fields F. The basic results are the proof of the uniqueness of a Whittaker vector, the existence of a Whittaker vector for cuspidal representations of $GL_n(F)$, the construction of an analog of the Gelfand-Tsetlin basis for cuspidal representations of $GL_n(F)$, and the description of cuspidal representations of $GL_n(F)$ in terms of Γ-functions [joint works with Graev and Kazhdan]. But I think that the most important work in this area is the complete description of irreducible representations of the groups $SL_2(F)$ and $GL_2(F)$ for local fields F with the residue characteristic different from 2 [a joint work with Graev and Kirillov]. They have shown that irreducible representations of $GL_2(F)$ are [essentially] parameterized by conjugacy classes of pairs (T, χ) where $T \subset GL_2(F)$ is a maximal torus and $\chi : T \to \mathbb{C}^*$ is a character. Moreover they found a formula for the characters $tr_{T\chi}(g)$ of these representations, and an explicit expression for the Plancherel measure. A striking and until now unexplained feature of these formulas is that they are essentially algebraic. For example, the Mellin transform [in χ] $L(g,t)$ of $tr_{T_\chi}(g)$ which is a function on $GL_2(F) \times T$ is given by

$$L(g,t) = \delta(det(g), \mathrm{Nm}(t))\epsilon_T(tr(g) - \mathbf{t}r(t))/|tr(g) - \mathbf{t}r(t)|.$$

Here $T = E^*$ where E is a quadratic extension of F, $\epsilon_T : F^* \to \pm$ is the quadratic character corresponding to E, tr is the matrix trace, and $\mathbf{t}r, \mathrm{Nm}$ are the trace and norm maps from E to F.

The understanding of the existence of an intrinsic connection between the structure of irreducible representations of groups over local fields and number theory was greatly clarified by Langlands. On the other hand, a generalization of algebraic formulas for the Mellin transform of characters and for the Plancherel measure was never found.

In an another work Gelfand and Graev found a description of the irreducible representations of the multiplicative group D^* of quaternions over F as induced from 1-dimensional representations of appropriate subgroups. This way to construct irreducible representations of D^* as induced from 1-dimensional representations of appropriate subgroups was later extended by R. Howe to other p-adic groups.

The description of representations of the groups $SL_2(F)$ and $GL_2(F)$ for local fields is presented n the book "Generalized functions v.6" [writtem with Graev and Piatetsky-Shapiro]. In the same book Gelfand developes the theroy of representations of semisimple adelic groups $G(\mathbb{A}_K)$ for global fields K. He defines the cuspidal part $L^2_0(G(\mathbb{A}_K)\backslash G(K) \subset L^2(G(\mathbb{A}_K)\backslash G(K)$ of the space of automorphic forms, proves that the representations of $G(\mathbb{A}_K)$ on $L^2_0(G(\mathbb{A}_K)\backslash G(K)$ is a direct sum of irreducible representations and develops a representation -theoretic interpretation of the theory of modular forms. The works of Langlands are very much influenced by these works of Gelfand.

It became clear that the description of a generic representations of any semisiple Lie group G [or a Lie algebra $\mathfrak{g}$] almost did not depend on a choice of particular group. So Gelfand tried to find a way to express this similarity in an intrisic way. In a series works with Kirillov he studies the structure of the skew-field $F(\mathfrak{g})$ of fractions for the universal envelopping algebra $U(\mathfrak{g})$. He found that the skew-fields $F(\mathfrak{g})$ are almost defined by the tansendence degree of the center $Z((\mathfrak{g}))$ [equal to the rank $r(\mathfrak{g})$ of $\mathfrak{g}$] and the their *Gelfand-Kirillov dimension* [equal to $1\backslash 2(dim(\mathfrak{g}) - r(\mathfrak{g}))$. These works are the basic of works of A. Joseph on the structure of the category of Harish-Chandra modules.

The last series of works of Gelfand on representation theory is on category $\mathcal{O}$ of representations of the Lie algebra $\mathfrak{g}$ of G. This category of representations was defined by Verma but the basic results are due to J. Bernstein, I. Gelfand, and S. Gelfand. They constructed a resolution of finite-dimensional representations by Verma modules $V_w, w \in W$ [known as the BGG resolution], discovered the duality between irreducible and projective modules in the category $\mathcal{O}$, and found the relation between the category $\mathcal{O}$ and the category of Harish-Chandra modules. These results are the corner stone of the theory of representations of semi-simple Lie algebras and their affine analogs.

But their main discovery is the existence of a strong connection between algebraic geometry of the flag space $\mathcal{B}$ and the structure of the category $\mathcal{O}$. For example, they have shown that there is an imbedding of V_w into $V_{w'}$ iff the Bruhat cell $BwB \subset \mathcal{B}$ is in the closure of $Bw'B$. This connection between algebraic geometry and the category of representations is the basis for the recent geometric theory of representations.

I did not discuss a number of other important works of Gelfand on representation theory [such as indecomposable representations of semisimple Lie groups, models of representations and representations of infinite-dimensional groups] but I want to mention two series of works which originated in the representation theory but have an independent life. The appearance of such works is very natural, since for Gelfand representation theory is a part of a much broader structure of analysis.

Integral geometry is an offshoot of the representation theory.The proof of the Plancherel theorem for complex groups is equivalent to the construction of the inversion formula which gives the value of a function in terms of its integrals over horocycles. Gelfand [in a series of joint works with Graev,Shapiro, Gindikin, Goncharov...] found inversion formulas for reconstruction of the value of a function on a manifold in terms of its integrals over an appropriate family of submanifolds. The existence of such inversion formulas found applications in such areas as symplectic geometry, multidimensional complex analysis, algebraic analysis, nonlinear differential equations, and aspects of Riemannian geometry, and also in applied mathematics (tomography).

Analogously, the works of Gelfand [with Ponomarev and later with Bernstein] on quivers were motivated by the problems of representations theory- the description of indecomposable representations for the Lorentz group. But the the inner development of the subject led to a beautiful and deep theory which later in works of Ringel, Lusztig and Nakajima made a full circle and became the foundation for the geometric representation theory of Lie algebras and quantum groups.