Spectral questions in endoscopic transfer for real reductive groups

Diana Shelstad

May 20, 2013
Introduction

a. endoscopic transfer vs stable transfer

- two related transfer principles introduced by Langlands 1970±, 2010±
Introduction

a. endoscopic transfer vs stable transfer

- two related transfer principles introduced by Langlands 1970±, 2010±
- archimedean local case and its relation to broader picture
Introduction

a. endoscopic transfer vs stable transfer

- two related transfer principles introduced by Langlands 1970±, 2010±
- archimedean local case and its relation to broader picture

- **endoscopic transfer** relates invariant harmonic analysis on given group $G(\mathbb{R})$ to stable harmonic analysis on the generally lower dimensional endoscopic groups $H_1(\mathbb{R})$
Introduction

a. endoscopic transfer vs stable transfer

- two related transfer principles introduced by Langlands 1970±, 2010±
- archimedean local case and its relation to broader picture

- endoscopic transfer relates invariant harmonic analysis on given group $G(\mathbb{R})$ to stable harmonic analysis on the generally lower dimensional endoscopic groups $H_1(\mathbb{R})$

- part of broader themes involving stable conjugacy, packets of representations and stabilization of the Arthur-Selberg trace formula
Introduction

a. endoscopic transfer vs stable transfer

- two related transfer principles introduced by Langlands 1970±, 2010±
- archimedean local case and its relation to broader picture

- **endoscopic transfer** relates invariant harmonic analysis on given group \( G(\mathbb{R}) \) to stable harmonic analysis on the generally lower dimensional endoscopic groups \( H_1(\mathbb{R}) \)

- part of broader themes involving stable conjugacy, packets of representations and stabilization of the Arthur-Selberg trace formula

- second principle, **stable transfer**, concerns stable harmonic analysis on any two groups \( G(\mathbb{R}), H(\mathbb{R}) \) related by a morphism of \( L \)-groups, part of *Beyond Endoscopy*, not discussed here
Introduction

b. endoscopic transfer: geometric side vs spectral side

- stable conjugacy in $G(\mathbb{R})$: $G(\mathbb{C})$-conjugacy with small refinement
stable conjugacy in $G(\mathbb{R})$: $G(\mathbb{C})$-conjugacy with small refinement

start with **geometric transfer**: unstable combinations of orbital integrals on given group $G(\mathbb{R})$ match stable combinations on an endoscopic group $H_1(\mathbb{R})$
Introduction

b. endoscopic transfer: geometric side vs spectral side

- stable conjugacy in $G(\mathbb{R})$: $G(\mathbb{C})$-conjugacy with small refinement

- start with **geometric transfer**: unstable combinations of orbital integrals on given group $G(\mathbb{R})$ match stable combinations on an endoscopic group $H_1(\mathbb{R})$

- matching: based on norm correspondence for very regular stable conjugacy classes in $H_1(\mathbb{R})$ and (twisted) classes in $G(\mathbb{R})$
stable conjugacy in $G(\mathbb{R})$: $G(\mathbb{C})$-conjugacy with small refinement

start with **geometric transfer**: unstable combinations of orbital integrals on given group $G(\mathbb{R})$ match stable combinations on an endoscopic group $H_1(\mathbb{R})$

matching: based on norm correspondence for very regular stable conjugacy classes in $H_1(\mathbb{R})$ and (twisted) classes in $G(\mathbb{R})$

matching provides a transfer of test functions from $G(\mathbb{R})$ to $H_1(\mathbb{R})$, then a dual map from 3-finite stable distributions on $H_1(\mathbb{R})$ to 3-finite invariant distributions on $G(\mathbb{R})$
b. endoscopic transfer: geometric side vs spectral side

- stable conjugacy in $G(\mathbb{R})$: $G(\mathbb{C})$-conjugacy with small refinement

- start with **geometric transfer**: unstable combinations of orbital integrals on given group $G(\mathbb{R})$ match stable combinations on an endoscopic group $H_1(\mathbb{R})$

- matching: based on norm correspondence for very regular stable conjugacy classes in $H_1(\mathbb{R})$ and (twisted) classes in $G(\mathbb{R})$

- matching provides a transfer of test functions from $G(\mathbb{R})$ to $H_1(\mathbb{R})$, then a dual map from $\mathbb{Z}$-finite stable distributions on $H_1(\mathbb{R})$ to $\mathbb{Z}$-finite invariant distributions on $G(\mathbb{R})$

- **spectral transfer**: interpret this dual map in terms of traces of irreducible admissible representations
c. our approach

- geometric side: transfer for orbital integrals has been proved using *transfer factors*
c. our approach

- geometric side: transfer for orbital integrals has been proved using \textit{transfer factors}.
- transfer factors = coefficients for unstable combinations: are defined \textit{a priori} and have various properties useful for descent arguments, comparison among inner forms, global questions \textit{etc.} \\
[Langlands-Shelstad, Kottwitz-Shelstad]
geometric side: transfer for orbital integrals has been proved using *transfer factors*

*transfer factors* = coefficients for unstable combinations: are defined *a priori* and have various properties useful for descent arguments, comparison among inner forms, global questions *etc.*

[Langlands-Shelstad, Kottwitz-Shelstad]

introduce spectral transfer factors with same basic structure (incomplete) and prove similar properties
geometric side: transfer for orbital integrals has been proved using *transfer factors*

transfer factors = coefficients for unstable combinations: are defined *a priori* and have various properties useful for descent arguments, comparison among inner forms, global questions *etc.*

[Langlands-Shelstad, Kottwitz-Shelstad]

introduce spectral transfer factors with same basic structure (incomplete) and prove similar properties

show that they are the only possible coefficients for spectral interpretation of dual transfer
c. our approach

- geometric side: transfer for orbital integrals has been proved using *transfer factors*

- transfer factors = coefficients for unstable combinations: are defined *a priori* and have various properties useful for descent arguments, comparison among inner forms, global questions etc. [Langlands-Shelstad, Kottwitz-Shelstad]

- introduce spectral transfer factors with same basic structure (incomplete) and prove similar properties

- show that they are the only possible coefficients for spectral interpretation of dual transfer

- apply this to various known identities to get (partial) spectral transfer
Introduction

c. our approach

- geometric side: transfer for orbital integrals has been proved using *transfer factors*
- transfer factors = coefficients for unstable combinations: are defined *a priori* and have various properties useful for descent arguments, comparison among inner forms, global questions *etc*.
  
  [Langlands-Shelstad, Kottwitz-Shelstad]
- introduce spectral transfer factors with same basic structure (incomplete) and prove similar properties
- show that they are the only possible coefficients for spectral interpretation of dual transfer
- apply this to various known identities to get (partial) spectral transfer
- the spectral factors contain precise information needed about packets
Endoscopic transfer: geometric side

a. general twisted setting

- $G$ connected, reductive algebraic group defined over $\mathbb{R}$
- $\theta$ an $\mathbb{R}$-automorphism of $G$, $\varpi$ a quasi-character on $G(\mathbb{R})$

study representations $\pi$ for which $\pi \circ \theta \simeq \varpi \otimes \pi$
Endoscopic transfer: geometric side

a. general twisted setting

- $G$ connected, reductive algebraic group defined over $\mathbb{R}$
  - $\theta$ an $\mathbb{R}$-automorphism of $G$, $\varpi$ a quasi-character on $G(\mathbb{R})$

study representations $\pi$ for which $\pi \circ \theta \simeq \varpi \otimes \pi$

- quasi-split data $(G^*, \theta^*)$:
  - $G^*$ quasi-split over $\mathbb{R}$, has an $\mathbb{R}$-splitting $spl^* = (B^*, T^*, \{X_\alpha\})$
    - [ultimately choice of $spl^*$ will not matter]
  - $\theta^*$ an $\mathbb{R}$-automorphism of $G^*$ preserving $spl^*$
Endoscopic transfer: geometric side

a. general twisted setting

- $G$ connected, reductive algebraic group defined over $\mathbb{R}$
- $\theta$ an $\mathbb{R}$-automorphism of $G$, $\varpi$ a quasi-character on $G(\mathbb{R})$
- study representations $\pi$ for which $\pi \circ \theta \simeq \varpi \otimes \pi$

- **quasi-split data** $(G^*, \theta^*)$:
  - $G^*$ quasi-split over $\mathbb{R}$, has an $\mathbb{R}$-splitting $spl^* = (B^*, T^*, \{X_\alpha\})$
  - [ultimately choice of $spl^*$ will not matter]
  - $\theta^*$ an $\mathbb{R}$-automorphism of $G^*$ preserving $spl^*$

- **inner form** $(G, \theta, \eta)$ of $(G^*, \theta^*)$:
  - $(G, \theta)$ as above, and $\eta : G \rightarrow G^*$ an inner twist such that
  - $\eta$ transports $\theta$ to $\theta^*$ up to inner automorphism:
    - $\theta = Int(h_\theta) \circ \eta^{-1} \circ \theta^* \circ \eta$, where $h_\theta \in G$
Endoscopic transfer: geometric side

a. general twisted setting

- $G$ connected, reductive algebraic group defined over $\mathbb{R}$
- $\theta$ an $\mathbb{R}$-automorphism of $G$, $\varpi$ a quasi-character on $G(\mathbb{R})$
- study representations $\pi$ for which $\pi \circ \theta \simeq \varpi \otimes \pi$

- quasi-split data $(G^*, \theta^*)$:
  - $G^*$ quasi-split over $\mathbb{R}$, has an $\mathbb{R}$-splitting $spl^* = (B^*, T^*, \{X_\alpha\})$
  - [ultimately choice of $spl^*$ will not matter]
  - $\theta^*$ an $\mathbb{R}$-automorphism of $G^*$ preserving $spl^*$

- inner form $(G, \theta, \eta)$ of $(G^*, \theta^*)$:
  - $(G, \theta)$ as above, and $\eta : G \rightarrow G^*$ an inner twist such that
  - $\eta$ transports $\theta$ to $\theta^*$ up to inner automorphism:
    $$\theta = Int(h_{\theta}) \circ \eta^{-1} \circ \theta^* \circ \eta,$$
    where $h_{\theta} \in G$

- up to isomorphism of inner forms, can arrange that transport $\eta^{-1} \circ \theta^* \circ \eta$ is defined over $\mathbb{R}$, so $Int(h_{\theta}) \in G_{ad}(\mathbb{R})$ [use fundamental splittings — exist for all $G$]
b. dual data

- **dual complex group** $G^\vee$ with splitting $s pl^\vee$ dual to $s pl^*$, action of Weil group $W_\mathbb{R}$ through $W_\mathbb{R} \rightarrow \Gamma = Gal(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$ action preserves $s pl^\vee$, and $L$-**group** $L^{} G = G^\vee \rtimes W_\mathbb{R}$
b. dual data

- **dual complex group** $G^\vee$ with splitting $spl^\vee$ dual to $spl^*$, action of Weil group $W_{\mathbb{R}}$ through $W_{\mathbb{R}} \to \Gamma = Gal(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$ action preserves $spl^\vee$, and $L$-group $^L G = G^\vee \rtimes W_{\mathbb{R}}$

- **automorphism** $\theta^\vee$ of $G^\vee$: preserves $spl^\vee$ and dual to $\theta^*$ quasi-character $\varpi$ comes from $a : W_{\mathbb{R}} \to ^L Z = Center(G^\vee) \rtimes W_{\mathbb{R}}$
b. dual data

- **dual complex group** $G^\vee$ with splitting $spl^\vee$ dual to $spl^*$, action of Weil group $W_\mathbb{R}$ through $W_\mathbb{R} \to \Gamma = \text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$ action preserves $spl^\vee$, and $L$-group $L G = G^\vee \rtimes W_\mathbb{R}$

- **automorphism** $\theta^\vee$ of $G^\vee$: preserves $spl^\vee$ and dual to $\theta^*$ quasi-character $\varpi$ comes from $a : W_\mathbb{R} \to L Z = \text{Center}(G^\vee) \rtimes W_\mathbb{R}$

- **automorphism** $L \theta_a$ of $L G$ extends $\theta^\vee$ with twist by $a$: $L \theta_a(g \times w) = \theta^\vee(g) a(w)$, for $g \in G^\vee, w \in W_\mathbb{R}$
**dual complex group** $G^\vee$ with splitting $spl^\vee$ dual to $spl^*$, action of Weil group $W_\mathbb{R}$ through $W_\mathbb{R} \to \Gamma = Gal(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$ action preserves $spl^\vee$, and $L$-group $L^G = G^\vee \rtimes W_\mathbb{R}$

**automorphism** $\theta^\vee$ of $G^\vee$: preserves $spl^\vee$ and dual to $\theta^*$ quasi-character $\varpi$ comes from $a : W_\mathbb{R} \to L^Z = Center(G^\vee) \rtimes W_\mathbb{R}$

**automorphism** $L\theta_a$ of $L^G$ extends $\theta^\vee$ with twist by $a$: $L\theta_a(g \times w) = \theta^\vee(g)a(w)$, for $g \in G^\vee$, $w \in W_\mathbb{R}$

in talk: assume $G^\vee$-component of $a$ is **bounded**, so $\varpi$ unitary [otherwise, insert *essentially* in various statements ...]
(bounded) supplemented endoscopic data \( \varepsilon_z \):
endoscopic data \( \varepsilon = (H, \mathcal{H}, s) \), together with
z-extension data \( (H_1, \xi_1) \)  
[Kaletha refinement ...]
(bounded) supplemented endoscopic data $\epsilon_z$:
endoscopic data $\epsilon = (\mathcal{H}, s)$, together with
$z$-extension data $(H_1, \xi_1)$  

[Kaletha refinement ...]

basic picture:

$$1 \rightarrow Cent_{\theta^\vee}(s, G_\vee)^0 \rightarrow \mathcal{H} \quad \Leftrightarrow \quad W_\mathbb{R} \rightarrow 1$$  \hspace{1cm} (1)$$

where $W_\mathbb{R}$ acts on $Cent_{\theta^\vee}(s, G_\vee)^0 = H_\vee$ by conjugation
by elements of $Cent_{\theta^\vee}(s, L G)$
in talk: assume $\theta$ preserves a fundamental splitting
[at each step should note effect of extra twist by elt of $G_{ad}(\mathbb{R})$]
in talk: assume $\theta$ preserves a fundamental splitting

[at each step should note effect of extra twist by elt of $G_{ad}(\mathbb{R})$]

there is $\Gamma$-map $\mathcal{A}$ from the set $Cl_{ss}(H_1)$ of semisimple conjugacy classes in $H_1(\mathbb{C})$ to the set $Cl_{\theta-ss}(G, \theta)$ of $\theta$-semisimple $\theta$-conjugacy classes in $G(\mathbb{C})$:

$$
\begin{align*}
Cl_{ss}(H_1) & \quad \downarrow \\
Cl_{ss}(H) & \quad \xrightarrow{\text{endo}} \quad Cl_{\theta^*-ss}(G^*, \theta) & \quad \xrightarrow{\text{inner}} \quad Cl_{\theta-ss}(G, \theta)
\end{align*}
$$

(2)
in talk: assume \( \theta \) **preserves a fundamental splitting**
[at each step should note effect of extra twist by elt of \( G_{ad}(\mathbb{R}) \)]

there is \( \Gamma \)-map \( \mathcal{A} \) from the set \( Cl_{ss}(H_1) \) of semisimple conjugacy classes in \( H_1(\mathbb{C}) \) to the set \( Cl_{\theta \cdot ss}(G, \theta) \) of \( \theta \)-semisimple \( \theta \)-conjugacy classes in \( G(\mathbb{C}) \):

\[
\begin{array}{ccc}
Cl_{ss}(H_1) & \xrightarrow{\text{endo}} & Cl_{\theta \cdot ss}(G^*, \theta) \\
\downarrow & & \downarrow \\
Cl_{ss}(H) & \xrightarrow{\text{inner}} & Cl_{\theta \cdot ss}(G, \theta)
\end{array}
\]  \hspace{1cm} (2)

\( \gamma_1 \) is **strongly \( G \)-regular** if and only if \( \mathcal{A} \) maps its class to a class of strongly \( \theta \)-regular elements in \( G \)
in talk: assume $\theta$ preserves a fundamental splitting
[at each step should note effect of extra twist by elt of $G_{ad}(\mathbb{R})$]

there is $\Gamma$-map $A$ from the set $Cl_{ss}(H_1)$ of semisimple conjugacy classes in $H_1(\mathbb{C})$ to the set $Cl_{\theta-ss}(G, \theta)$ of $\theta$-semisimple $\theta$-conjugacy classes in $G(\mathbb{C})$:

$$ Cl_{ss}(H_1) \downarrow \quad Cl_{ss}(H) \xrightarrow{\text{endo}} \quad Cl_{\theta^*\text{-ss}}(G^*, \theta) \xrightarrow{\text{inner}} \quad Cl_{\theta\text{-ss}}(G, \theta) $$ (2)

$\gamma_1$ is strongly $G$-regular if and only if $A$ maps its class to a class of strongly $\theta$-regular elements in $G$

strongly $G$-regular $\gamma_1$ is a norm of strongly $\theta$-regular $\delta$, i.e. $(\gamma_1, \delta)$ is a norm pair, if and only if $\delta$ is in image of class of $\gamma_1$
sufficient to specify geometric transfer on **very regular set**: all pairs \((\gamma_1, \delta) \in H_1(\mathbb{R}) \times G(\mathbb{R})\), where \(\gamma_1\) is strongly \(G\)-regular and \(\delta\) is strongly \(\theta\)-regular
sufficient to specify geometric transfer on very regular set:
all pairs $(\gamma_1, \delta) \in H_1(\mathbb{R}) \times G(\mathbb{R})$, where $\gamma_1$ is strongly $G$-regular and $\delta$ is strongly $\theta$-regular

**transfer factor** $\Delta$ is complex-valued function on very regular set
Endoscopic transfer: geometric side

d. transfer factors

- sufficient to specify geometric transfer on **very regular set**: all pairs $(\gamma_1, \delta) \in H_1(\mathbb{R}) \times G(\mathbb{R})$, where $\gamma_1$ is strongly $G$-regular and $\delta$ is strongly $\theta$-regular

- **transfer factor** $\Delta$ is complex-valued function on very regular set

- define $\Delta(\gamma_1, \delta) = 0$ if $(\gamma_1, \delta)$ is not a norm pair
Endoscopic transfer: geometric side

d. transfer factors

- sufficient to specify geometric transfer on **very regular set**: all pairs \((\gamma_1, \delta) \in H_1(\mathbb{R}) \times G(\mathbb{R})\), where \(\gamma_1\) is strongly \(G\)-regular and \(\delta\) is strongly \(\theta\)-regular

- **transfer factor** \(\Delta\) is complex-valued function on very regular set

- define \(\Delta(\gamma_1, \delta) = 0\) if \((\gamma_1, \delta)\) is not a norm pair

- now assume \((\gamma_1, \delta), (\gamma'_1, \delta')\) are norm pairs
sufficient to specify geometric transfer on **very regular set**: all pairs \((\gamma_1, \delta) \in H_1(\mathbb{R}) \times G(\mathbb{R})\), where \(\gamma_1\) is strongly \(G\)-regular and \(\delta\) is strongly \(\theta\)-regular

**transfer factor** \(\Delta\) is complex-valued function on very regular set

define \(\Delta(\gamma_1, \delta) = 0\) if \((\gamma_1, \delta)\) is not a norm pair

now assume \((\gamma_1, \delta), (\gamma'_1, \delta')\) are norm pairs

our transfer statement will not fix normalization for \(\Delta(\gamma_1, \delta)\) instead define canonical relative factor \(\Delta(\gamma_1, \delta; \gamma'_1, \delta')\) and use any factor \(\Delta(\gamma_1, \delta)\) satisfying

\[
\Delta(\gamma_1, \delta) / \Delta(\gamma'_1, \delta') = \Delta(\gamma_1, \delta; \gamma'_1, \delta')
\] (3)
sufficient to specify geometric transfer on **very regular set**: all pairs \((\gamma_1, \delta) \in H_1(\mathbb{R}) \times G(\mathbb{R})\), where \(\gamma_1\) is strongly \(G\)-regular and \(\delta\) is strongly \(\theta\)-regular

**transfer factor** \(\Delta\) is complex-valued function on very regular set

define \(\Delta(\gamma_1, \delta) = 0\) if \((\gamma_1, \delta)\) is not a norm pair

now assume \((\gamma_1, \delta), (\gamma'_1, \delta')\) are norm pairs

our transfer statement will not fix normalization for \(\Delta(\gamma_1, \delta)\) instead define canonical relative factor \(\Delta(\gamma_1, \delta; \gamma'_1, \delta')\) and use any factor \(\Delta(\gamma_1, \delta)\) satisfying

\[
\Delta(\gamma_1, \delta)/\Delta(\gamma'_1, \delta') = \Delta(\gamma_1, \delta; \gamma'_1, \delta')
\]

(3)

two versions of transfer: here use factors for classical version other version: (turns out to be) complex conjugate
definitions allow simultaneously treatment of inner forms
extended group = $K$-group: fills out stable conjugacy classes
• definitions allow simultaneously treatment of inner forms
  extended group = $K$-group: fills out stable conjugacy classes
• particular normalizations, esp. Whittaker normalization for
  several inner forms of quasi-split data $(G^*, \theta^*)$
Endoscopic transfer: geometric side

definitions allow simultaneously treatment of inner forms
extended group = \( K \)-group: fills out stable conjugacy classes

particular normalizations, esp. Whittaker normalization for
several inner forms of quasi-split data \((G^* , \theta^*)\)

relative \( \Delta \) is product \( \Delta_I \Delta_{II} \Delta_{III} \); only \( \Delta_{III} \) is genuinely relative
Endoscopic transfer: geometric side

definitions allow simultaneously treatment of inner forms
extended group = $K$-group: fills out stable conjugacy classes

particular normalizations, esp. Whittaker normalization for
several inner forms of quasi-split data $(G^*, \theta^*)$

relative $\Delta$ is product $\Delta_I \Delta_{II} \Delta_{III}$; only $\Delta_{III}$ is genuinely relative

$\Delta_I, \Delta_{III}$ have Galois-cohomological definitions,
spectral versions in same groups [sample at end of talk]
definitions allow simultaneously treatment of inner forms
extended group = $K$-group: fills out stable conjugacy classes

particular normalizations, esp. Whittaker normalization for
several inner forms of quasi-split data $(G^*, \theta^*)$

relative $\Delta$ is product $\Delta_I \Delta_{II} \Delta_{III}$; only $\Delta_{III}$ is genuinely relative
$\Delta_I, \Delta_{III}$ have Galois-cohomological definitions,
spectral versions in same groups [sample at end of talk]

$\Delta_{II}(\gamma_1, \delta)$ comes from analysis of jumps in orbital integrals
spectral version: different form, involves character formula
toral data associated with norm pair \((\gamma_1, \delta)\): there is \(\theta^*\)-stable pair \((B, T)\) in \(G^*\), with \(T\) defined over \(\mathbb{R}\), and various maps yielding

\[
\begin{align*}
\delta & \overset{inner}{\sim} \quad \delta^* \in T(\mathbb{C}) \\
\gamma_1 \overset{z}{\longrightarrow} \gamma_H \overset{endo}{\longleftarrow} \gamma^* \in T_{\theta^*}(\mathbb{R})
\end{align*}
\]  
(4)
• toral data associated with norm pair \((\gamma_1, \delta)\): there is \(\theta^*\)-stable pair \((B, T)\) in \(G^*\), with \(T\) defined over \(\mathbb{R}\), and various maps yielding

\[
\delta \overset{\text{inner}}{\sim} \delta^* \in T(\mathbb{C}) \\
\downarrow \\
\gamma_1 \xrightarrow{z} \gamma_H \xrightarrow{\text{endo}} \gamma^* \in T_{\theta^*}(\mathbb{R})
\]  

\(4\)

\(R_{res} = \theta^*\)-restricted root system for \(T\) in \(G^*\), Galois orbits \(O_{res}\)
\(R_1 = \text{root system for } T_1 \text{ in } H_1, \text{ Galois orbits } O_1\)

to each indivisible \(O_{res}\) attach well-defined \(\chi_{\alpha}(\frac{N_{\alpha}(\delta^*)r_{\alpha}}{a_{\alpha}} - 1)\)
to each \(O_1\) attach well-defined \(\chi_{\alpha_1}(\frac{\alpha_1(\gamma_1) - 1}{a_{\alpha_1}})\) \([\text{notation}]\)
\(\Delta_{II}(\gamma_1, \delta)\) is quotient over all indivisible \(O_{res}\) by all \(O_1\)
**Endoscopic transfer: geometric side**

ddd. transfer factors (cont.)

- **toral data** associated with norm pair \((\gamma_1, \delta)\): there is \(\theta^*\)-stable pair \((B, T)\) in \(G^*\), with \(T\) defined over \(\mathbb{R}\), and various maps yielding

\[
\begin{align*}
\delta & \overset{inner}{\sim} \delta^* \in T(\mathbb{C}) \\
\gamma_1 & \overset{z}{\rightarrow} \gamma_H
\end{align*}
\]

\[
\gamma^* \in T_{\theta^*}(\mathbb{R})
\]

- \(R_{res} = \theta^*\)-restricted root system for \(T\) in \(G^*\), Galois orbits \(O_{res}\)
- \(R_1 = \text{root system for } T_1 \text{ in } H_1\), Galois orbits \(O_1\)

  to each indivisible \(O_{res}\) attach well-defined \(\chi_\alpha\left(\frac{N_\alpha(\delta^*)^{r_\alpha} - 1}{a_\alpha}\right)\)

  to each \(O_1\) attach well-defined \(\chi_\alpha_1\left(\frac{\alpha_1(\gamma_1) - 1}{a_\alpha_1}\right)\) [notation]

\(\Delta_{II}(\gamma_1, \delta)\) is quotient over all indivisible \(O_{res}\) by all \(O_1\)

- **\(\chi\)-data, \(a\)-data:** \(\{\chi_\alpha\}\), \(\{a_\alpha\}\) etc. as above
**Endoscopic transfer: geometric side**

**ddd. transfer factors (cont.)**

- **toral data** associated with norm pair \((\gamma_1, \delta)\): there is \(\theta^*\)-stable pair \((B, T)\) in \(G^*\), with \(T\) defined over \(\mathbb{R}\), and various maps yielding

\[
\delta \xrightarrow{\text{inner}} \delta^* \in T(C)
\]

\[
\begin{array}{c}
\gamma_1 \xrightarrow{z} \gamma_H \xrightarrow{\text{endo}} \gamma^* \in T_{\theta^*}(\mathbb{R})
\end{array}
\]

- \(R_{res} = \theta^*\)-restricted root system for \(T\) in \(G^*\), Galois orbits \(O_{res}\)
  - \(R_1 = \) root system for \(T_1\) in \(H_1\), Galois orbits \(O_1\)
  - to each indivisible \(O_{res}\) attach well-defined \(\chi_\alpha \left( \frac{N_\alpha(\delta^*)^{r_\alpha} - 1}{a_\alpha} \right)\)
  - to each \(O_1\) attach well-defined \(\chi_{\alpha_1} \left( \frac{\alpha_1(\gamma_1)^{\gamma_1} - 1}{a_{\alpha_1}} \right)\) [notation]
  - \(\Delta_{II}(\gamma_1, \delta)\) is quotient over all indivisible \(O_{res}\) by all \(O_1\)

- **\(\chi\)-data, \(a\)-data:** \(\{\chi_\alpha\}, \{a_\alpha\}\) etc. as above

- same data used in \(\Delta_I, \Delta_{III}\); two of the three affect each of relative \(\Delta_I, \Delta_{II}, \Delta_{III}\) but product \(\Delta\) is independent of all choices
Endoscopic transfer: geometric side

e. main theorem and corollary [Sh 2012]

Theorem

For each $\theta$-Schwartz fdg on $G(\mathbb{R})$ there exists $\lambda_1$-Schwartz $f_1 dh_1$ on $H_1(\mathbb{R})$ such that

$$SO(\gamma_1, f_1 dh_1) = \sum_{\delta, \theta-\text{conj}} \Delta(\gamma_1, \delta) \cdot O^{\theta,\infty}(\delta, \text{fdg})$$

(5)

for all strongly $G$-regular $\gamma_1$ in $H_1(\mathbb{R})$.

Corollary

If $f$ has compact support then we may take $f_1$ of compact support mod $Z_1(\mathbb{R})$. 
corollary follows immediately from a theorem of Bouaziz
corollary follows immediately from a theorem of Bouaziz

notation: \( Z_1 = \text{Ker}(H_1 \rightarrow H) \), \( \varepsilon_z \) determines character \( \lambda_1 \) on \( Z_1(\mathbb{R}) \), require \( f_1(z_1 h_1) = \lambda_1(z_1)^{-1} f_1(h_1) \) for \( z_1 \in Z_1(\mathbb{R}), h_1 \in H_1(\mathbb{R}) \)
corollary follows immediately from a theorem of Bouaziz

notation: $Z_1 = \ker(H_1 \to H)$, $\varepsilon_z$ determines character $\lambda_1$ on $Z_1(\mathbb{R})$, require $f_1(z_1 h_1) = \lambda_1(z_1)^{-1} f_1(h_1)$ for $z_1 \in Z_1(\mathbb{R})$, $h_1 \in H_1(\mathbb{R})$

$\Delta(\gamma_1, \delta)$ is invariant under stable conjugacy in first variable, also has correct behavior under translation by $Z_1(\mathbb{R})$
corollary follows immediately from a theorem of Bouaziz.

notation: $Z_1 = \text{Ker}(H_1 \to H)$, $\varepsilon_z$ determines character $\lambda_1$ on $Z_1(\mathbb{R})$, require $f_1(z_1 h_1) = \lambda_1(z_1)^{-1} f_1(h_1)$ for $z_1 \in Z_1(\mathbb{R}), h_1 \in H_1(\mathbb{R})$.

$\Delta(\gamma_1, \delta)$ is invariant under stable conjugacy in first variable, also has correct behavior under translation by $Z_1(\mathbb{R})$.

$SO(\gamma_1, f_1 dh_1)$ is usual normalized stable orbital integral.
corollary follows immediately from a theorem of Bouaziz

notation: $Z_1 = \text{Ker}(H_1 \to H)$, $\varepsilon_z$ determines character $\lambda_1$ on $Z_1(\mathbb{R})$, require $f_1(z_1 h_1) = \lambda_1(z_1)^{-1} f_1(h_1)$ for $z_1 \in Z_1(\mathbb{R})$, $h_1 \in H_1(\mathbb{R})$

$\Delta(\gamma_1, \delta)$ is invariant under stable conjugacy in first variable, also has correct behavior under translation by $Z_1(\mathbb{R})$

$SO(\gamma_1, f_1 dh_1)$ is usual normalized stable orbital integral

left and right: compatible Haar measures in denominators of quotients
corollary follows immediately from a theorem of Bouaziz

notation: \( Z_1 = \text{Ker}(H_1 \to H) \), \( \epsilon_z \) determines character \( \lambda_1 \) on \( Z_1(\mathbb{R}) \), require \( f_1(z_1 h_1) = \lambda_1(z_1)^{-1} f_1(h_1) \) for \( z_1 \in Z_1(\mathbb{R}) \), \( h_1 \in H_1(\mathbb{R}) \)

\( \Delta(\gamma_1, \delta) \) is invariant under stable conjugacy in first variable, also has correct behavior under translation by \( Z_1(\mathbb{R}) \)

\( SO(\gamma_1, f_1 dh_1) \) is usual normalized stable orbital integral

left and right: compatible Haar measures in denominators of quotients

\( (\theta, \varpi) \)-twisted orbital integral

\[
O^{\theta,\varpi}(\delta, fdg) := \int_{\text{Cent}_{\theta}(\delta, G)(\mathbb{R}) \backslash G(\mathbb{R})} f(g^{-1} \delta \theta(g)) \varpi(g) \frac{dg}{dt_\delta} \quad (6)
\]
corollary follows immediately from a theorem of Bouaziz

notation: \( Z_1 = \text{Ker}(H_1 \to H) \), \( \varepsilon_z \) determines character \( \lambda_1 \) on \( Z_1(\mathbb{R}) \), require \( f_1(z_1 h_1) = \lambda_1(z_1)^{-1} f_1(h_1) \) for \( z_1 \in Z_1(\mathbb{R}), h_1 \in H_1(\mathbb{R}) \)

\( \Delta(\gamma_1, \delta) \) is invariant under stable conjugacy in first variable, also has correct behavior under translation by \( Z_1(\mathbb{R}) \)

\( SO(\gamma_1, f_1 dh_1) \) is usual normalized stable orbital integral

left and right: compatible Haar measures in denominators of quotients

\((\theta, \omega)\)-twisted orbital integral

\[
O^{\theta, \omega}(\delta, f dg) := \int_{\text{Cent}_\theta(\delta, G)(\mathbb{R}) \setminus G(\mathbb{R})} f(g^{-1} \delta \theta(g)) \omega(g) \frac{dg}{dt_\delta} \quad (6)
\]

\( \Delta(\gamma_1, \delta) \) has correct behavior under \( \theta \)-conjugacy to make right side of (5) well-defined
For proof of theorem:
For proof of theorem:

- (old) characterization of stable orbital integrals via Harish-Chandra Plancherel theory in terms of jump behavior
For proof of theorem:

(1) characterization of stable orbital integrals via Harish-Chandra Plancherel theory in terms of jump behavior

introduce form better adapted to canonical transfer factors
For proof of theorem:

(old) characterization of stable orbital integrals via Harish-Chandra Plancherel theory in terms of jump behavior

introduce form better adapted to canonical transfer factors

Harish-Chandra descent for twisted orbital integrals and *semi-regular is sufficient* principle, along with descent properties of the norm correspondence, reduce problem to simple wall-crossing properties for transfer factors
For proof of theorem:

- (old) characterization of stable orbital integrals via Harish-Chandra Plancherel theory in terms of jump behavior
- introduce form better adapted to canonical transfer factors

- Harish-Chandra descent for twisted orbital integrals and *semi-regular is sufficient* principle, along with descent properties of the norm correspondence, reduce problem to simple wall-crossing properties for transfer factors

- (long) calculations with transfer factors to check these properties
for each test $fdg$ on $G(\mathbb{R})$ attach test $f_1 dh_1$ on $H_1(\mathbb{R})$
with matching orbital integrals in the sense of (5) of main theorem
for each test $fg$ on $G(\mathbb{R})$ attach test $f_1 dh_1$ on $H_1(\mathbb{R})$
with matching orbital integrals in the sense of (5) of main theorem

$\Theta_1$ : stable distribution on $H_1(\mathbb{R})$, correct $Z_1(\mathbb{R})$ behavior
and eigendistribution for center $Z_1$ of universal enveloping algebra
for each test $fdg$ on $G(\mathbb{R})$ attach test $f_1 dh_1$ on $H_1(\mathbb{R})$
with matching orbital integrals in the sense of (5) of main theorem

$\Theta_1$ : stable distribution on $H_1(\mathbb{R})$, correct $Z_1(\mathbb{R})$ behavior
and eigendistribution for center $Z_1$ of universal enveloping algebra

then $\Theta : fdg \to \Theta_1(f_1 dh_1)$ well-defined $\theta$-twisted invariant
distribution on $G(\mathbb{R})$ and eigendistribution for $Z_1$
for each test $fdg$ on $G(\mathbb{R})$ attach test $f_1 dh_1$ on $H_1(\mathbb{R})$
with matching orbital integrals in the sense of (5) of main theorem

$\Theta_1$ : stable distribution on $H_1(\mathbb{R})$, correct $Z_1(\mathbb{R})$ behavior
and eigendistribution for center $Z_1$ of universal enveloping algebra

then $\Theta : fdg \rightarrow \Theta_1(f_1 dh_1)$ well-defined $\theta$-twisted invariant
distribution on $G(\mathbb{R})$ and eigendistribution for $\mathfrak{Z}$

$\Theta_1$ tempered $\implies \Theta$ tempered
Endoscopic transfer: spectral side

a. dual transfer: summary

- for each test $fdg$ on $G(\mathbb{R})$ attach test $f_1dh_1$ on $H_1(\mathbb{R})$ with matching orbital integrals in the sense of (5) of main theorem

- $\Theta_1$: stable distribution on $H_1(\mathbb{R})$, correct $Z_1(\mathbb{R})$ behavior and eigendistribution for center $Z_1$ of universal enveloping algebra

- then $\Theta : fdg \rightarrow \Theta_1(f_1dh_1)$ well-defined $\theta$-twisted invariant distribution on $G(\mathbb{R})$ and eigendistribution for $Z$

- $\Theta_1$ tempered $\implies \Theta$ tempered

- endo $\varepsilon_z$ determines shift in infinitesimal character
Endoscopic transfer: spectral side

a. dual transfer: summary

- for each test $fdg$ on $G(\mathbb{R})$ attach test $f_1dh_1$ on $H_1(\mathbb{R})$
  with matching orbital integrals in the sense of (5) of main theorem

- $\Theta_1$: stable distribution on $H_1(\mathbb{R})$, correct $Z_1(\mathbb{R})$ behavior
  and eigendistribution for center $Z_1$ of universal enveloping algebra

- then $\Theta: fdg \rightarrow \Theta_1(f_1dh_1)$ well-defined $\theta$-twisted invariant
  distribution on $G(\mathbb{R})$ and eigendistribution for $Z$

- $\Theta_1$ tempered $\implies\Theta$ tempered

- endo $\varepsilon_z$ determines shift in infinitesimal character

- formula for $\Theta_1$ as smooth function on regular set
  $\implies$ formula for $\Theta$ as smooth function on regular set
**goal:** for a stable character $\Theta_1 = \text{St} \cdot \text{Trace} \pi_1$, where $\pi_1$ irreducible admissible representation of $H_1(\mathbb{R})$ with correct $Z_1(\mathbb{R})$ behavior, to describe $\Theta$ explicitly as a combination of $(\theta, \omega)$-twisted traces

$$f \rightarrow \text{Trace} [\pi(f) \circ \pi(\theta, \omega)] \quad (7)$$

notation: $\pi(\theta, \omega)$ intertwines $\pi \circ \theta$ and $\omega \otimes \pi$ [also drop $dg, dh$]
Endoscopic transfer: spectral side

b. dual transfer as spectral transfer

- **goal:** for a stable character $\Theta_1 = St-Trace \pi_1$, where $\pi_1$ irreducible admissible representation of $H_1(\mathbb{R})$ with correct $Z_1(\mathbb{R})$ behavior, to describe $\Theta$ explicitly as a combination of $(\theta, \omega)$-twisted traces

$$ f \rightarrow Trace [\pi(f) \circ \pi(\theta, \omega)] $$  \hspace{1cm} (7)

notation: $\pi(\theta, \omega)$ intertwines $\pi \circ \theta$ and $\omega \otimes \pi$ \hspace{1cm} [also drop dg, dh]

- **thus to establish** dual transfer in the form

$$ St-Trace \pi_1(f_1) = \sum_{\pi} \Delta(\pi_1, \pi) \left[ Trace \pi(f) \circ \pi(\theta, \omega) \right] $$  \hspace{1cm} (8)
goal: for a stable character $\Theta_1 = St-\text{Trace} \, \pi_1$, where $\pi_1$ irreducible admissible representation of $H_1(\mathbb{R})$ with correct $Z_1(\mathbb{R})$ behavior, to describe $\Theta$ explicitly as a combination of $(\theta, \varpi)$-twisted traces

$$f \mapsto \text{Trace} \left[ \pi(f) \circ \pi(\theta, \varpi) \right]$$

(7)

notation: $\pi(\theta, \varpi)$ intertwines $\pi \circ \theta$ and $\varpi \otimes \pi$ [also drop $dg, dh$]

thus to establish dual transfer in the form

$$St-\text{Trace} \, \pi_1(f_1) = \sum_{\pi} \Delta(\pi_1, \pi) \left[ \text{Trace} \, \pi(f) \circ \pi(\theta, \varpi) \right]$$

(8)

term on right side will be independent of normalization of $\pi(\theta, \varpi)$ [$\Delta_{II}$ involves twisted character formula and effects cancel]
in place of very regular norm pairs \((\gamma_1, \delta), (\gamma_1', \delta')\),
consider very regular related pairs \((\pi_1, \pi), (\pi_1', \pi')\):
define (almost) canonical \(\Delta(\pi_1, \pi; \pi_1', \pi')\)
in place of very regular norm pairs $(\gamma_1, \delta), (\gamma'_1, \delta')$, consider very regular related pairs $(\pi_1, \pi), (\pi'_1, \pi')$:
define (almost) canonical $\Delta(\pi_1, \pi; \pi'_1, \pi')$

again $\Delta$ has same form $\Delta_I \Delta_{II} \Delta_{III}$; may also define $\Delta(\pi_1, \pi; \gamma_1, \delta)$
in place of very regular norm pairs \((\gamma_1, \delta), (\gamma'_1, \delta')\),
consider very regular related pairs \((\pi_1, \pi), (\pi'_1, \pi')\):
deﬁne (almost) canonical \(\Delta(\pi_1, \pi; \pi'_1, \pi')\)

again \(\Delta\) has same form \(\Delta_I \Delta_{II} \Delta_{III}\); may also deﬁne \(\Delta(\pi_1, \pi; \gamma_1, \delta)\)
in transfer theorems use geom-spec compatible factors:
\[\Delta(\pi_1, \pi)/\Delta(\gamma_1, \delta) = \Delta(\pi_1, \pi; \gamma_1, \delta)\]
in place of very regular norm pairs \((\gamma_1, \delta), (\gamma'_1, \delta')\),
consider very regular related pairs \((\pi_1, \pi), (\pi'_1, \pi')\):
define (almost) canonical \(\Delta(\pi_1, \pi; \pi'_1, \pi')\)

again \(\Delta\) has same form \(\Delta_I, \Delta_{II}, \Delta_{III}\); may also define \(\Delta(\pi_1, \pi; \gamma_1, \delta)\)
in transfer theorems use geom-spec compatible factors:
\[\Delta(\pi_1, \pi)/\Delta(\gamma_1, \delta) = \Delta(\pi_1, \pi; \gamma_1, \delta)\]

**standard setting:** \(\theta = identity, \varpi = trivial character\)
results \(\implies\) structure on packets of representations
... then twisted setting \(\implies\) compatible additional structure
on packets preserved by \(\pi \mapsto \varpi^{-1} \otimes (\pi \circ \theta)\)
Endoscopic transfer: spectral side

c. very regular pairs

- prescribe very regular pairs via Arthur parameters, start with $G^*$
Endoscopic transfer: spectral side

c. very regular pairs

- prescribe very regular pairs via Arthur parameters, start with $G^*$

- Arthur parameter: $G \cap$-conjugacy class of an admissible hom

$$\psi = (\varphi, \rho) : W_\mathbb{R} \times SL(2, \mathbb{C}) \to L^1 G$$

here $\varphi$ [in general, essentially] bounded Langlands parameter

$\psi_0$ central $\rho(\Sl(2, \mathbb{C}))$ in $L G$

$M_\psi = M_\varphi$ in $L G$ as subgp gen by $M_\varphi$ and $\varphi(W_\mathbb{R})^{-1} M_\varphi W_\mathbb{R}^{-1}$

extract $L$-action same way as endo, $M$ in dual, quasi-split over $\mathbb{R}$
Endoscopic transfer: spectral side

c. very regular pairs

- prescribe very regular pairs via Arthur parameters, start with $G^*$

- Arthur parameter: $G^\vee$-conjugacy class of an admissible hom
  \[ \psi = (\varphi, \rho) : W_\mathbb{R} \times SL(2, \mathbb{C}) \to {}^L G \]
  here $\varphi$ [in general, essentially] bounded Langlands parameter

- let $S = S_\psi = Cent(\psi(W_\mathbb{R} \times SL(2, \mathbb{C})), G^\vee)$: $\psi$ is elliptic if $S^0$ central
c. very regular pairs

- prescribe very regular pairs via Arthur parameters, start with $G^*$

- **Arthur parameter:** $G^\vee$-conjugacy class of an admissible hom
  \[
  \psi = (\varphi, \rho) : \mathcal{W}_\mathbb{R} \times SL(2, \mathbb{C}) \to L^G
  \]
  here $\varphi$ [in general, essentially] bounded Langlands parameter

- let $S = S_\psi = Cent(\psi(W_\mathbb{R} \times SL(2, \mathbb{C})), G^\vee)$: $\psi$ is **elliptic** if $S^0$
  central

- $\rho(SL(2, \mathbb{C})) \subset M^\vee = M^\vee_\varphi = \text{Levi group } Cent(\varphi(\mathbb{C}^\times), G^\vee)$ in $G^\vee$
Endoscopic transfer: spectral side

\textbf{c. very regular pairs}

- prescribe very regular pairs via Arthur parameters, start with $G^*$

- Arthur parameter: $G^\vee$-conjugacy class of an admissible hom
  \[ \psi = (\varphi, \rho) : \mathcal{W}_\mathbb{R} \times SL(2, \mathbb{C}) \rightarrow \mathcal{L}G \]
  here $\varphi$ [in general, essentially] bounded Langlands parameter

- let $S = S_\psi = \text{Cent}(\psi(\mathcal{W}_\mathbb{R} \times SL(2, \mathbb{C})), G^\vee)$: $\psi$ is \textbf{elliptic} if $S^0$
  central

- $\rho(SL(2, \mathbb{C})) \subset M^\vee = M^\vee_\varphi = \text{Levi group Cent}(\varphi(\mathbb{C}^\times), G^\vee)$ in $G^\vee$

- call $\psi$ \textbf{u-regular} if $\rho(SL(2, \mathbb{C}))$ contains regular unipotent elts of $M^\vee$
Endoscopic transfer: spectral side

c. very regular pairs

- prescribe very regular pairs via Arthur parameters, start with $G^*$

- Arthur parameter: $G^\vee$-conjugacy class of an admissible hom
  \[ \psi = (\varphi, \rho) : W_{\mathbb{R}} \times SL(2, \mathbb{C}) \to ^L G \]
  here $\varphi$ [in general, essentially] bounded Langlands parameter

- let $S = S_\psi = \text{Cent}(\psi(W_{\mathbb{R}} \times SL(2, \mathbb{C})), G^\vee)$: $\psi$ is \textbf{elliptic} if $S^0$ central

- $\rho(SL(2, \mathbb{C})) \subset M^\vee = M_\varphi^\vee = \text{Levi group } \text{Cent}(\varphi(\mathbb{C}^\times), G^\vee)$ in $G^\vee$

- call $\psi$ \textbf{u-regular} if $\rho(SL(2, \mathbb{C}))$ contains regular unipotent elts of $M^\vee$

- define group $\mathcal{M} = \mathcal{M}_\varphi$ in $^L G$ as subgp gen by $M^\vee$ and $\varphi(W_{\mathbb{R}})$
  \[ 1 \to M^\vee \to \mathcal{M} \subseteq W_{\mathbb{R}} \to 1 \]
  extract $L$-action same way as endo, $\mathcal{M}^* =$ dual, quasi-split over $\mathbb{R}$
Endoscopic transfer: spectral side
cc. very regular pairs (cont.)

- $u$-regular $\psi$ is elliptic $\iff T \hookrightarrow M^* \hookrightarrow G^*$ all over $\mathbb{R}$, with $T$ anisotropic modulo the center of $G$
Endoscopic transfer: spectral side
cc. very regular pairs (cont.)

- $u$-regular $\psi$ is elliptic $\iff T \hookrightarrow M^* \hookrightarrow G^*$ all over $\mathbb{R}$, with $T$ anisotropic modulo the center of $G$
- $u$-regular $\psi = (\varphi, \text{triv})$ is elliptic $\iff \varphi$ discrete series parameter
Endoscopic transfer: spectral side

cc. very regular pairs (cont.)

- \( u \)-regular \( \psi \) is elliptic \iff \( T \hookrightarrow M^* \hookrightarrow G^* \) all over \( \mathbb{R} \), with \( T \) anisotropic modulo the center of \( G \)
- \( u \)-regular \( \psi = (\varphi, \text{triv}) \) is elliptic \iff \( \varphi \) discrete series parameter
- attach packet \( \Pi \) to \( u \)-regular \( \psi : L \)-packet if \( \rho = \text{triv} \), or Arthur packet otherwise [see Adams-Johnson, just elliptic here]
**Endoscopic transfer: spectral side**

**cc. very regular pairs (cont.)**

- $u$-regular $\psi$ is elliptic $\iff T \hookrightarrow M^* \hookrightarrow G^*$ all over $\mathbb{R}$, with $T$ anisotropic modulo the center of $G$

- $u$-regular $\psi = (\varphi, \text{triv})$ is elliptic $\iff \varphi$ discrete series parameter

- Attach packet $\Pi$ to $u$-regular $\psi$ : $L$-packet if $\rho = \text{triv}$, or Arthur packet otherwise [see Adams-Johnson, just elliptic here]

- Do same for endo group: use only those $u$-regular $\psi_1$ such that $\psi_1(W_{\mathbb{R}} \times SL(2, \mathbb{C}))$ lies in the image of endo $\mathcal{H}$, up to conjugacy [$\iff$ members of attached $\Pi_1$ have correct $Z_1(\mathbb{R})$ behavior]
Endoscopic transfer: spectral side
cc. very regular pairs (cont.)

- $u$-regular $\psi$ is elliptic \iff $T \hookrightarrow M^* \hookrightarrow G^*$ all over $\mathbb{R}$, with $T$ anisotropic modulo the center of $G$
- $u$-regular $\psi = (\varphi, \text{triv})$ is elliptic \iff $\varphi$ discrete series parameter

attach packet $\Pi$ to $u$-regular $\psi : L$-packet if $\rho = \text{triv}$, or Arthur packet otherwise [see Adams-Johnson, just elliptic here]

do same for endo group: use only those $u$-regular $\psi_1$ such that $\psi_1(\mathcal{W}_\mathbb{R} \times SL(2, \mathbb{C}))$ lies in the image of endo $\mathcal{H}$, up to conjugacy [\iff members of attached $\Pi_1$ have correct $Z_1(\mathbb{R})$ behavior]

such $\psi_1$ determines parameter $\psi_{\psi_1}$ for $G^*$, Levi group $M_1$ for $\psi_1$ determines subgroup $M_H$ of $\mathcal{H}$ contained in Levi $M$ for $\psi_{\psi_1} :$ call $\psi_1$ $G$-regular if $M_H = M$
Endoscopic transfer: spectral side

cc. very regular pairs (cont.)

- $u$-regular $\psi$ is elliptic $\iff T \hookrightarrow M^* \hookrightarrow G^*$ all over $\mathbb{R}$, with $T$ anisotropic modulo the center of $G$
- $u$-regular $\psi = (\varphi, \text{triv})$ is elliptic $\iff \varphi$ discrete series parameter
- attach packet $\Pi$ to $u$-regular $\psi : L$-packet if $\rho = \text{triv}$, or Arthur packet otherwise [see Adams-Johnson, just elliptic here]
- do same for endo group: use only those $u$-regular $\psi_1$ such that $\psi_1(\mathcal{W}_\mathbb{R} \times \text{SL}(2, \mathbb{C}))$ lies in the image of endo $\mathcal{H}$, up to conjugacy [$\iff$ members of attached $\Pi_1$ have correct $Z_1(\mathbb{R})$ behavior]
- such $\psi_1$ determines parameter $\psi_{\psi_1}$ for $G^*$, Levi group $\mathcal{M}_1$ for $\psi_1$ determines subgroup $\mathcal{M}_H$ of $\mathcal{H}$ contained in Levi $\mathcal{M}$ for $\psi_{\psi_1} :$ call $\psi_1$ $G$-regular if $\mathcal{M}_H = \mathcal{M}$
- $(\psi_1, \psi)$ very regular pair: $\psi_1, \psi$ are $u$-regular and $\psi_1$ is $G$-regular
Endoscopic transfer: spectral side
cc. very regular pairs (cont.)

- $u$-regular $\psi$ is elliptic $\iff T \hookrightarrow M^* \hookrightarrow G^*$ all over $\mathbb{R}$, with $T$ anisotropic modulo the center of $G$
- $u$-regular $\psi = (\varphi, \text{triv})$ is elliptic $\iff \varphi$ discrete series parameter
- attach packet $\Pi$ to $u$-regular $\psi : L$-packet if $\rho = \text{triv}$, or Arthur packet otherwise [see Adams-Johnson, just elliptic here]
- do same for endo group: use only those $u$-regular $\psi_1$ such that $\psi_1(\mathcal{W}_\mathbb{R} \times SL(2,\mathbb{C}))$ lies in the image of endo $\mathcal{H}$, up to conjugacy [$\iff$ members of attached $\Pi_1$ have correct $Z_1(\mathbb{R})$ behavior]
- such $\psi_1$ determines parameter $\psi_{\psi_1}$ for $G^*$, Levi group $M_1$ for $\psi_1$ determines subgroup $M_H$ of $\mathcal{H}$ contained in Levi $M$ for $\psi_{\psi_1}$ : call $\psi_1$ $G$-regular if $M_H = M$
- $(\psi_1, \psi)$ very regular pair: $\psi_1, \psi$ are $u$-regular and $\psi_1$ is $G$-regular
- very regular related pair: also $\psi = \psi_{\psi_1}$
same defs for pairs $(\pi_1, \pi)$ in packets $(\Pi_1, \Pi)$ attached to $(\psi_1, \psi)$
same defs for pairs \((\pi_1, \pi)\) in packets \((\Pi_1, \Pi)\) attached to \((\psi_1, \psi)\)

start with standard setting, tempered \((\rho = \text{triv})\) and elliptic:

(8) says: \(St\text{-}Trace \pi_1(f_1) = \sum_{\pi} \Delta(\pi_1, \pi) \text{Trace} \pi(f)\)

\((\pi_1, \pi), (\pi'_1, \pi')\) related pairs discrete series representations
with Langlands parameters \((\varphi_1, \varphi), (\varphi'_1, \varphi')\)
Endoscopic transfer: spectral side

d. standard setting: tempered pairs

- **same defs for** pairs $(\pi_1, \pi)$ in packets $(\Pi_1, \Pi)$ attached to $(\psi_1, \psi)$

- **start with standard setting, tempered** $(\rho = \text{triv})$ **and elliptic:**
  
  (8) says: $St\text{-}Trace \pi_1(f_1) = \sum_{\pi} \Delta(\pi_1, \pi) \text{Trace} \pi(f)$

  $(\pi_1, \pi), (\pi'_1, \pi')$ related pairs discrete series representations with Langlands parameters $(\varphi_1, \varphi), (\varphi'_1, \varphi')$

- define relative factor $\Delta(\pi_1, \pi; \pi'_1, \pi')$
- **same def** for pairs $(\pi_1, \pi)$ in packets $(\Pi_1, \Pi)$ attached to $(\psi_1, \psi)$

- **start with standard setting, tempered** $(\rho = \text{triv})$ and elliptic: (8) says: $St\text{-}Trace \pi_1(f_1) = \sum_{\pi} \Delta(\pi_1, \pi) \text{Trace} \pi(f)$

  $(\pi_1, \pi), (\pi'_1, \pi')$ related pairs discrete series representations with Langlands parameters $(\varphi_1, \varphi), (\varphi'_1, \varphi')$

- define relative factor $\Delta(\pi_1, \pi; \pi'_1, \pi')$

- toral data $T_1 \to T$, with $T$ anisotropic mod center of $G$, $a$-data, $\chi$-data for $\Delta_l, \Delta_{ll}, \Delta_{lll}$
Endoscopic transfer: spectral side

d. standard setting: tempered pairs

- **same defs for** pairs \((\pi_1, \pi)\) in packets \((\Pi_1, \Pi)\) attached to \((\psi_1, \psi)\)

- **start with standard setting, tempered** \((\rho = \text{triv})\) and **elliptic**: (8) says: \(\text{St-Trace } \pi_1(f_1) = \sum_{\pi} \Delta(\pi_1, \pi) \text{Trace } \pi(f)\)

  \((\pi_1, \pi), (\pi'_1, \pi')\) related pairs discrete series representations with Langlands parameters \((\varphi_1, \varphi), (\varphi'_1, \varphi')\)

- define relative factor \(\Delta(\pi_1, \pi; \pi'_1, \pi')\)

- toral data \(T_1 \to T\), with \(T\) anisotropic mod center of \(G\), a-data, \(\chi\)-data for \(\Delta_I, \Delta_{II}, \Delta_{III}\)

- \(\Delta_{II}\) involves local formula for \(\text{Trace } \pi(f)\) as smooth function ...

[fourth root of unity if rewrite usual Harish-Chandra formula]
via parabolic induction extend defns to $\Delta(\pi_1, \pi; \pi'_1, \pi')$, $\Delta(\pi_1, \pi; \gamma_1, \delta)$, for all very regular norm pairs $(\gamma_1, \delta)$ and all tempered very regular related pairs $(\pi_1, \pi), (\pi'_1, \pi')$ [set $\Delta(\pi_1, \pi) = 0$ if pair not related]
via parabolic induction extend defns to $\Delta(\pi_1, \pi; \pi'_1, \pi')$, $\Delta(\pi_1, \pi; \gamma_1, \delta)$, for all very regular norm pairs $(\gamma_1, \delta)$ and all tempered very regular related pairs $(\pi_1, \pi), (\pi'_1, \pi')$ [set $\Delta(\pi_1, \pi) = 0$ if pair not related]

**proof of (8) for tempered very regular pairs:** reduce quickly to elliptic case, discrete series both sides, and then apply Harish-Chandra characterization theorem: transfer $\Theta$ is tempered invariant eigendistribution with correct infinitesimal character and agrees with $\sum_{\pi} \Delta(\pi_1, \pi) \text{Trace } \pi(f)$ on regular elliptic set
via parabolic induction extend defns to $\Delta(\pi_1, \pi; \pi'_1, \pi')$, $\Delta(\pi_1, \pi; \gamma_1, \delta)$, for all very regular norm pairs $(\gamma_1, \delta)$ and all tempered very regular related pairs $(\pi_1, \pi), (\pi'_1, \pi')$

[set $\Delta(\pi_1, \pi) = 0$ if pair not related]

**proof of (8) for tempered very regular pairs:** reduce quickly to elliptic case, discrete series both sides, and then apply Harish-Chandra characterization theorem: transfer $\Theta$ is tempered invariant eigendistribution with correct infinitesimal character and agrees with $\sum_{\pi} \Delta(\pi_1, \pi) \text{Trace } \pi(f)$ on regular elliptic set

**now theorem for all tempered pairs?** for example, need this for converse: spec transfer for $(f_1, f) \implies$ geom transfer for $(f_1, f)$
main case = elliptic on left: transfer discrete series to limits of discrete series, limits which arise have Levi $\mathcal{M}$ of type $(A_1)^n$ then Hecht-Schmid character identities + analysis in $G^\vee$ identifies transfer $\Theta$ as right side of (8), where factor $\Delta(\pi_1, \pi)$ is defined via analog of Zuckerman translation for parameters
Endoscopic transfer: spectral side

e. standard setting: tempered transfer theorem

- main case = elliptic on left: transfer discrete series to limits of discrete series, limits which arise have Levi $\mathcal{M}$ of type $(A_1)^n$
  then Hecht-Schmid character identities + analysis in $G^\vee$ identifies transfer $\Theta$ as right side of (8), where factor $\Delta(\pi_1, \pi)$ is defined via analog of Zuckerman translation for parameters

- conclude the following continuation of geom transfer thm, std setting:
Endoscopic transfer: spectral side

• main case = elliptic on left: transfer discrete series to limits of discrete series, limits which arise have Levi $\mathcal{M}$ of type $(A_1)^n$ then Hecht-Schmid character identities + analysis in $G^\vee$ identifies transfer $\Theta$ as right side of (8), where factor $\Delta(\pi_1, \pi)$ is defined via analog of Zuckerman translation for parameters

• conclude the following continuation of geom transfer thm, std setting:

Theorem

Suppose geom, spec factors $\Delta$ are compatible. Then

$$\text{St-Trace } \pi_1(f_1 dh_1) = \sum_{\pi} \Delta(\pi_1, \pi) \text{Trace } \pi(fdg)$$

for all tempered irreducible admissible representations $\pi_1$ such that $Z_1(\mathbb{R})$ acts by $\lambda_1$. 
Conversely: if \( fdg, f_1 dh_1 \) are test measures satisfying (9) then

\[
SO(\gamma_1, f_1 dh_1) = \sum_{\delta \text{ conj}} \Delta(\gamma_1, \delta) O(\delta, fdg)
\]  

for all strongly \( G \)-regular \( \gamma_1 \) in \( Z_1(\mathbb{R}) \).

Proof: Use both transfer theorems plus same \( SO' \)s \( \implies \) same \( St \)-Traces
Conversely: if $fdg$, $f_1 dh_1$ are test measures satisfying \( (9) \) then
\[
SO(\gamma_1, f_1 dh_1) = \sum_{\delta \text{ conj}} \Delta(\gamma_1, \delta) \, O(\delta, fdg)
\] (10)

for all strongly $G$-regular $\gamma_1$ in $Z_1(\mathbb{R})$.

Proof: Use both transfer thms plus same $SO$'s $\implies$ same $St$-Traces

**alternate argument** to prove tempered spectral transfer:
Conversely: if $fdg, f_1dh_1$ are test measures satisfying (9) then

$$SO(\gamma_1, f_1dh_1) = \sum_{\delta \text{ conj}} \Delta(\gamma_1, \delta) O(\delta, fdg)$$

for all strongly $G$-regular $\gamma_1$ in $Z_1(\mathbb{R})$.

Proof: Use both transfer thms plus same $SO$'s $\implies$ same $St$-Traces

 alternate argument to prove tempered spectral transfer:

(i) in the elliptic case the chosen $\Delta(\pi_1, \pi)$ are the only possible coefficients for a spectral version of dual transfer ..., plus they have correct properties re translation principle and parabolic induction ... again this depends also on properties of the geometric factors and compatibility factors
Conversely: if $fdg$, $f_1dh_1$ are test measures satisfying (9) then

$$SO(\gamma_1, f_1dh_1) = \sum_{\delta \text{ conj}} \Delta(\gamma_1, \delta) O(\delta, fdg)$$

for all strongly $G$-regular $\gamma_1$ in $Z_1(R)$. Proof: Use both transfer thms plus same $SO$’s $\implies$ same $St$-Traces.

alternate argument to prove tempered spectral transfer:

(i) in the elliptic case the chosen $\Delta(\pi_1, \pi)$ are the only possible coefficients for a spectral version of dual transfer ..., plus they have correct properties re translation principle and parabolic induction ... again this depends also on properties of the geometric factors and compatibility factors

(ii) theorem is true for some choice of coefficients [old result] and so it is true with the factors $\Delta(\pi_1, \pi)$ we have defined
still in standard setting, nontempered examples?
define $\Delta(\pi_1, \pi)$ for very regular pairs in general:

enough to define $\Delta(\pi_1, \pi; \pi'_1, \pi')$ for some tempered $(\pi'_1, \pi')$,
then $\Delta(\pi_1, \pi) := \Delta(\pi_1, \pi; \pi'_1, \pi').\Delta(\pi'_1, \pi')$. 
Endoscopic transfer: spectral side

g. standard setting: very regular pairs in general

still in standard setting, nontempered examples?
define $\Delta(\pi_1, \pi)$ for very regular pairs in general:
enough to define $\Delta(\pi_1, \pi; \pi'_1, \pi')$ for some tempered $(\pi'_1, \pi')$,
then $\Delta(\pi_1, \pi) := \Delta(\pi_1, \pi; \pi'_1, \pi').\Delta(\pi'_1, \pi')$

start with elliptic case: construct $(\pi'_1, \pi')$ tempered elliptic
or just $\pi'_1$ tempered elliptic in some cases
still in standard setting, nontempered examples? 
define $\Delta(\pi_1, \pi)$ for very regular pairs in general: 
   enough to define $\Delta(\pi_1, \pi; \pi_1', \pi')$ for some tempered $(\pi_1', \pi')$, 
   then $\Delta(\pi_1, \pi) := \Delta(\pi_1, \pi; \pi_1', \pi').\Delta(\pi_1', \pi')$

start with elliptic case: construct $(\pi_1', \pi')$ tempered elliptic 
   or just $\pi_1'$ tempered elliptic in some cases

for transfer statement (9): apply alternate argument to 
character identities of Adams-Johnson [see Arthur, Kottwitz]
still in standard setting, nontempered examples? define $\Delta(\pi_1, \pi)$ for very regular pairs in general: enough to define $\Delta(\pi_1, \pi; \pi'_1, \pi')$ for some tempered $(\pi'_1, \pi')$, then $\Delta(\pi_1, \pi) := \Delta(\pi_1, \pi; \pi'_1, \pi').\Delta(\pi'_1, \pi')$

start with elliptic case: construct $(\pi'_1, \pi')$ tempered elliptic or just $\pi'_1$ tempered elliptic in some cases

for transfer statement (9): apply alternate argument to character identities of Adams-Johnson [see Arthur, Kottwitz]

or check directly that these factors $\Delta(\pi_1, \pi)$ work in A-J arguments: use familiar formula for relative factor $\Delta(\pi_1, \pi; \pi'_1, \pi') := \Delta(\pi_1, \pi)/\Delta(\pi'_1, \pi')$ when $\pi, \pi'$ lie in same Arthur packet
still in standard setting, nontempered examples?
define $\Delta(\pi_1, \pi)$ for very regular pairs in general:
enough to define $\Delta(\pi_1, \pi; \pi'_1, \pi')$ for some tempered $(\pi'_1, \pi')$,
then $\Delta(\pi_1, \pi) := \Delta(\pi_1, \pi; \pi'_1, \pi').\Delta(\pi'_1, \pi')$

start with elliptic case: construct $(\pi'_1, \pi')$ tempered elliptic
or just $\pi'_1$ tempered elliptic in some cases

for transfer statement (9): apply alternate argument to
character identities of Adams-Johnson [see Arthur, Kottwitz]

or check directly that these factors $\Delta(\pi_1, \pi)$ work in A-J
arguments: use familiar formula for relative factor
$\Delta(\pi_1, \pi; \pi'_1, \pi') := \Delta(\pi_1, \pi)/\Delta(\pi'_1, \pi')$ when $\pi, \pi'$ lie in same
Arthur packet

[remove elliptic assumption]
return to twist by \((\theta, \varpi)\) and start with tempered setting
h. general twisted setting

- return to twist by \((\theta, \omega)\) and start with tempered setting

- now concerned only with \((\theta, \omega)\)-stable packets \(\Pi\), \(i.e.\) those \(\Pi\) preserved by \(\pi \rightarrow \omega^{-1} \otimes (\pi \circ \theta)\), along with attached twist-packet \(\Pi'^{\theta, \omega}\) consisting of those \(\pi \in \Pi\) fixed by this map
return to twist by $(\theta, \wp)$ and start with tempered setting

now concerned only with $(\theta, \wp)$-stable packets $\Pi$, i.e. those $\Pi$ preserved by $\pi \mapsto \wp^{-1} \otimes (\pi \circ \theta)$, along with attached twist-packet $\Pi^{\theta,\wp}$ consisting of those $\pi \in \Pi$ fixed by this map

enough: $\theta$ preserves fundamental splitting [earlier comment]
return to twist by $(\theta, \varpi)$ and start with tempered setting

now concerned only with $(\theta, \varpi)$-stable packets $\Pi$, i.e. those $\Pi$ preserved by $\pi \rightarrow \varpi^{-1} \otimes (\pi \circ \theta)$, along with attached twist-packet $\Pi^{\theta, \varpi}$ consisting of those $\pi \in \Pi$ fixed by this map

enough: $\theta$ preserves fundamental splitting [earlier comment]

essentially harmonic analysis on group $\mathcal{G}(\mathbb{R}) \rtimes \langle \theta \rangle$ outside Harish-Chandra class [some results not yet written in sufficient generality to claim transfer results in general]
return to twist by \((\theta, \varpi)\) and start with tempered setting

now concerned only with \((\theta, \varpi)\)-stable packets \(\Pi\), \textit{i.e.} those \(\Pi\) preserved by \(\pi \rightarrow \varpi^{-1} \otimes (\pi \circ \theta)\), along with attached twist-packet \(\Pi^{\theta, \varpi}\) consisting of those \(\pi \in \Pi\) fixed by this map

enough: \(\theta\) preserves fundamental splitting [earlier comment]

essentially harmonic analysis on group \(G(\mathbb{R}) \rtimes \langle \theta \rangle\) outside Harish-Chandra class [some results not yet written in sufficient generality to claim transfer results in general]

approach to defining tempered spectral factors: again elliptic setting first, translation, and then parabolic descent [Mezo 2013: use results of Duflo for parabolic induction]
spectral factors in tempered elliptic case: now constructions parallel those for twisted geometric factors of Kottwitz-Shelstad, again compatibility factors, parallel properties, etc.
spectral factors in tempered elliptic case: now constructions parallel those for twisted geometric factors of Kottwitz-Shelstad, again compatibility factors, parallel properties, etc.

Proof of transfer: apply alternate argument again, here to character identities of Mezo
Endoscopic transfer: spectral side

hh. general twisted setting (cont.)

- spectral factors in tempered elliptic case: now constructions parallel those for twisted geometric factors of Kottwitz-Shelstad, again compatibility factors, parallel properties, etc.

- Proof of transfer: apply alternate argument again, here to character identities of Mezo

- Mezo 2012: identities for elliptic $(\pi_1, \pi)$, also when only $\pi_1$ elliptic, with coefficients written in terms of data from Duflo’s method rather than directly from Harish-Chandra character formula
spectral factors in tempered elliptic case: now constructions parallel those for twisted geometric factors of Kottwitz-Shelstad, again compatibility factors, parallel properties, etc.

Proof of transfer: apply alternate argument again, here to character identities of Mezo

Mezo 2012: identities for elliptic \((\pi_1, \pi)\), also when only \(\pi_1\) elliptic, with coefficients written in terms of data from Duflo’s method rather than directly from Harish-Chandra character formula

again similar approach to standard case to define twisted factors \(\Delta(\pi_1, \pi)\) for nontempered very regular pairs \((\pi_1, \pi)\) ...
summary: along with geometric transfer factors come spectral factors, in both standard and twisted settings; these express dual transfer as a spectral transfer [incomplete ...]
summary: along with geometric transfer factors come spectral factors, in both standard and twisted settings; these express dual transfer as a spectral transfer [incomplete ...]

now we use the relative factors $\Delta(\pi_1, \pi; \pi'_1, \pi')$ to establish pairings of a packet $\Pi$ with a finite group defined on dual side
summary: along with geometric transfer factors come spectral factors, in both standard and twisted settings; these express dual transfer as a spectral transfer

now we use the relative factors $\Delta(\pi_1, \pi; \pi_1', \pi')$ to establish pairings of a packet $\Pi$ with a finite group defined on dual side

then twisted relative factors $\Delta(\pi_1, \pi; \pi_1', \pi')$ provide compatible pairings for twist-packets $\Pi^{\theta, \omega}$ within $(\theta, \omega)$-stable $\Pi$
summary: along with geometric transfer factors come spectral factors, in both standard and twisted settings; these express dual transfer as a spectral transfer

now we use the relative factors $\Delta(\pi_1, \pi; \pi'_1, \pi')$ to establish pairings of a packet $\Pi$ with a finite group defined on dual side

then twisted relative factors $\Delta(\pi_1, \pi; \pi'_1, \pi')$ provide compatible pairings for twist-packets $\Pi^{\theta,\omega}$ within $(\theta, \omega)$-stable $\Pi$

various (Galois-cohomological) properties of pairings have consequences for harmonic analysis, e.g. inversion of spectral transfer in tempered setting

[Whittaker normalizations $\implies$ simplest spectral pairings]
start with tempered packet $\Pi$ and use relative factors $\Delta(\pi_1, \pi; \pi_1, \pi')$, with $\pi, \pi' \in \Pi$, to put structure on $\Pi$
Structure on packets

a. standard setting

- start with tempered packet $\Pi$ and use relative factors $\Delta(\pi_1, \pi; \pi_1, \pi')$, with $\pi, \pi' \in \Pi$, to put structure on $\Pi$
- $\pi_1$ determined by spectral construction of endo data:
Structure on packets

a. standard setting

- start with tempered packet $\Pi$ and use relative factors $\Delta(\pi_1, \pi; \pi_1, \pi')$, with $\pi, \pi' \in \Pi$, to put structure on $\Pi$

- $\pi_1$ determined by spectral construction of endo data:

- $\phi : W_{IR} \to ^LG$ Langlands parameter for $\Pi$
  $S = Cent(\phi(W_{IR}), G^\vee)^0$, $S^{ad} = \text{image of } S \text{ in } (G^\vee)_{ad}$,
  $S^{sc} = \text{preimage of } S^{ad} \text{ in } (G^\vee)_{sc}$, $s_{sc} = \text{semisimple element in } S^{sc}$
start with tempered packet $\Pi$ and use relative factors $\Delta(\pi_1, \pi; \pi_1, \pi')$, with $\pi, \pi' \in \Pi$, to put structure on $\Pi$

$\pi_1$ determined by spectral construction of endo data:

$\varphi : W_{\mathbb{IR}} \to ^L G$ Langlands parameter for $\Pi$
$S = \text{Cent}(\varphi(W_{\mathbb{IR}}), G^\vee)^0, \quad S^{ad} = \text{image of } S \text{ in } (G^\vee)_{ad},$
$S^{sc} = \text{preimage of } S^{ad} \text{ in } (G^\vee)_{sc}, \quad s_{sc} = \text{semisimple element in } S^{sc}$

$s = \text{image of } s_{sc} \text{ in } G^\vee$
$\mathcal{H}(s) = \text{subgroup of } ^L G \text{ generated by } \text{Cent}(s, G^\vee)^0 \text{ and } \varphi(W_{\mathbb{IR}})$
$\varepsilon_z(s_{sc}) = \varepsilon_z(s) = \text{attached suppl. endo data}$
Structure on packets

a. standard setting

- start with tempered packet $\Pi$ and use relative factors $\Delta(\pi_1, \pi; \pi_1, \pi')$, with $\pi, \pi' \in \Pi$, to put structure on $\Pi$
- $\pi_1$ determined by spectral construction of endo data:

  - $\varphi : W_\mathbb{R} \to {}^L G$ Langlands parameter for $\Pi$
  - $S = \text{Cent}(\varphi(W_\mathbb{R}), G^\vee)^0$, $S^{ad} = \text{image of } S \text{ in } (G^\vee)^{ad}$,
  - $S^{sc} = \text{preimage of } S^{ad} \text{ in } (G^\vee)^{sc}$, $s_{sc} = \text{semisimple element in } S^{sc}$
  - $s = \text{image of } s_{sc} \text{ in } G^\vee$
  - $\mathcal{H}(s) = \text{subgroup of } {}^L G \text{ generated by } Cent(s, G^\vee)^0 \text{ and } \varphi(W_\mathbb{R})$
  - $\varepsilon_Z(s_{sc}) = \varepsilon_Z(s) = \text{attached suppl. endo data}$
  - by construction, $\varphi$ factors through well-positioned $\varphi^s : W_\mathbb{R} \to {}^L H_1$
Structure on packets

a. standard setting

- start with tempered packet $\Pi$ and use relative factors $\Delta(\pi_1, \pi; \pi_1, \pi')$, with $\pi, \pi' \in \Pi$, to put structure on $\Pi$
- $\pi_1$ determined by spectral construction of endo data:

  - $\varphi : W_\mathbb{IR} \rightarrow L^* G$ Langlands parameter for $\Pi$
  - $S = \text{Cent}(\varphi(W_\mathbb{IR}), G^\vee)^0$, $S^{ad} =$ image of $S$ in $(G^\vee)^{ad}$,
  - $S^{sc} =$ preimage of $S^{ad}$ in $(G^\vee)^{sc}$, $s^{sc} =$ semisimple element in $S^{sc}$
  - $s =$ image of $s^{sc}$ in $G^\vee$
  - $\mathcal{H}(s) =$ subgroup of $L^* G$ generated by $\text{Cent}(s, G^\vee)^0$ and $\varphi(W_\mathbb{IR})$
  - $\varepsilon_Z(s^{sc}) = \varepsilon_Z(s) =$ attached suppl. endo data
  - by construction, $\varphi$ factors through well-positioned $\varphi^s : W_\mathbb{IR} \rightarrow L^* H_1$
  - now for $\pi_1$ take any $\pi^s \in \Pi^s =$ packet attached to $\varphi^s$
Structure on packets

aa. standard setting

**Theorem:** \( s_{sc} \rightarrow \Delta(\pi^s, \pi; \pi^s, \pi') \) depends only on the image of \( s_{sc} \) under \( S^{sc} \rightarrow S^{ad} \rightarrow \pi_0(S^{ad}) = S^{ad} = \text{sum of } \mathbb{Z}/2's \)
Structure on packets

aa. standard setting

• **Theorem:** \( s_{sc} \rightarrow \Delta(\pi^s, \pi; \pi^s, \pi') \) depends only on the image of \( s_{sc} \) under \( S^{sc} \rightarrow S^{ad} \rightarrow \pi_0(S^{ad}) = S^{ad} = \text{sum of } \mathbb{Z}/2\text{'s} \)

• and defines character on \( S^{ad} \), trivial iff \( \pi = \pi' \), all ...

  [in general this requires a dual, uniform by packet, version of Knapp-Zuckerman decomposition of unitary principal series]
Structure on packets

aa. standard setting

- **Theorem:** $s_{sc} \to \Delta(\pi^s, \pi; \pi^s, \pi')$ depends only on the image of $s_{sc}$ under $S^{sc} \to S^{ad} \to \pi_0(S^{ad}) = S^{ad} = \text{sum of } \mathbb{Z}/2\text{'s}$

- and defines character on $S^{ad}$, trivial iff $\pi = \pi'$, all ...
  [in general this requires a dual, uniform by packet, version of Knapp-Zuckerman decomposition of unitary principal series]

- elliptic case: just Tate-Nakayama duality $\mathbb{C}/\mathbb{R}$
Theorem: \( s_{sc} \rightarrow \Delta(\pi^s, \pi; \pi^s, \pi') \) depends only on the image of \( s_{sc} \) under \( S^{sc} \rightarrow S^{ad} \rightarrow \pi_0(S^{ad}) = S^{ad} = \text{sum of } \mathbb{Z}/2's \)

- and defines character on \( S^{ad} \), trivial iff \( \pi = \pi' \), all ...
  
  [in general this requires a dual, uniform by packet, version of Knapp-Zuckerman decomposition of unitary principal series]

- elliptic case: just Tate-Nakayama duality \( \mathbb{C}/\mathbb{R} \)

- in general, don’t use duality with \( S^{ad} \) but with extension, e.g. \( S^{sc} \)
  
  so will write \( \Delta(\pi^s, \pi; \pi^s, \pi') = \langle \pi, s_{sc} \rangle / \langle \pi', s_{sc} \rangle \):
  
  pick base point \( \pi_0 \) for \( \Pi \) and specify character \( s_{sc} \rightarrow \langle \pi_0, s_{sc} \rangle \), then \( \langle \pi, s_{sc} \rangle := \Delta(\pi^s, \pi; \pi^s, \pi_0) \langle \pi_0, s_{sc} \rangle \)
  
  ... pairing of type proposed by Arthur for global picture [2007]
  
  [better, new approach of Kaletha]
**Theorem:** $s_{sc} \rightarrow \Delta(\pi^s, \pi; \pi^s, \pi^\prime)$ depends only on the image of $s_{sc}$ under $s^{sc} \rightarrow S^{ad} \rightarrow \pi_0(S^{ad}) = S^{ad} = \text{sum of } \mathbb{Z}/2\text{'s}$

and defines character on $S^{ad}$, trivial iff $\pi = \pi^\prime$, all ...

[in general this requires a dual, uniform by packet, version of Knapp-Zuckerman decomposition of unitary principal series]

**elliptic case:** just Tate-Nakayama duality $\mathbb{C}/\mathbb{R}$

**in general,** don’t use duality with $S^{ad}$ but with extension, e.g. $s^{sc}$ so will write $\Delta(\pi^s, \pi; \pi^s, \pi^\prime) = \langle \pi, s_{sc} \rangle / \langle \pi^\prime, s_{sc} \rangle$:

pick base point $\pi_0$ for $\Pi$ and specify character $s_{sc} \rightarrow \langle \pi_0, s_{sc} \rangle$,

then $\langle \pi, s_{sc} \rangle := \Delta(\pi^s, \pi; \pi^s, \pi_0) \langle \pi_0, s_{sc} \rangle$

... pairing of type proposed by Arthur for global picture [2007]

[better, new approach of Kaletha]

**simpler case... Theorem:** $G$ of quasi-split type, Whittaker norm of absolute $\Delta$, $\pi_0$ generic, trivial character $s_{sc} \rightarrow \langle \pi_0, s_{sc} \rangle$:

$\langle \pi, s \rangle := \Delta(\pi^s, \pi)$ gives perfect pairing ... $\Pi$ as dual of $S^{ad}$
**Corollary:** invert transfer in Whittaker setting simply as

$$Trace \; \pi(f) = \left| S^{ad} \right|^{-1} \sum_{s \in S^{ad}} \langle \pi, s \rangle \; St-Trace \; \pi^s(f_1^s) \quad (11)$$

for all tempered $\pi$, test $f$ and corresponding test $f_1^s$
**Corollary:** invert transfer in Whittaker setting simply as

\[
\text{Trace } \pi(f) = \left| S^{ad} \right|^{-1} \sum_{s \in S^{ad}} \langle \pi, s \rangle \text{ St-Trace } \pi^s(f_1^s)
\]  

(11)

for all tempered \( \pi \), test \( f \) and corresponding test \( f_1^s \)

- now review some constructions, focus on Whittaker case, and move to twisted setting ...
**Corollary:** invert transfer in Whittaker setting simply as

\[
\text{Trace } \pi(f) = \left| S^{ad} \right|^{-1} \sum_{s \in S^{ad}} \langle \pi, s \rangle \text{ St-Trace } \pi^{s}(f^{s})
\]  

(11)

for all tempered \( \pi \), test \( f \) and corresponding test \( f^{s} \)

- now review some constructions, focus on Whittaker case, and move to twisted setting ...
- elliptic case, Whittaker setting: calculate \( \langle \pi, s \rangle ? \) 
  
  \( G^{\ast} \) cuspidal, \( T \) anisotropic mod center, also \( T_{G} \subseteq G \)
  
  \( \pi = \text{ discrete series, } \pi_{0} \) determines Weyl chamber(s) \( C_{0} \)
  
  yielding toral data for \( T \) in \( G^{\ast} \) and then well-defined character \( \kappa \) on \( H^{1}(\Gamma, T^{sc}) \); \( \pi \) determines chamber for \( T_{G} \); inner twist carries this chamber to \( C_{0} \) up to inner automorphism; make a well defined element \( \omega \) in \( H^{1}(\Gamma, T^{sc}) \); finally, \( \langle \pi, s \rangle = \kappa(\omega) \)
c. twisted setting

- remarks on last calculation:

\[ h, s_i \text{ is the absolute version of } \Delta_{\text{III}} \text{ available in this setting} \]

In general setting, there is a central obstruction to defining \( \omega \) in \( H_1 \) which is handled by going to relative version using a trick from the original definition of geometric factors in [L-S].

This trick works for any pair \( T, T_0 \) of maximal tori over local field \( F \)...

- nonabelian variant for general elliptic \( u \)-regular case

- it is easy to extend this type of calculation (for discrete series) to the twisted setting using fundamental splittings (Weyl chambers find splittings):

  - assume \( \theta \) preserves find splitting \( \text{spl} \);
  - may assume inner twist \( \eta \) transports \( \text{spl} \) to find splitting \( \text{spl} \) of \( G \) preserved by \( \theta \),
  - \( \text{spl} \) provides toral data to transport objects from \( G \).
Structure on packets

c. twisted setting

- remarks on last calculation:
- (i) $\langle \pi, s \rangle$ is the absolute version of $\Delta_{\text{III}}$ available in this setting in general setting, there is a central obstruction to defining $\omega$ in $H^1$ which is handled by going to relative version using a trick from the original definition of geometric factors in [L-S] [trick works for any pair $T, T'$ of maximal tori over local field $F$ ...] nonabelian variant for general elliptic $u$-regular case
Structure on packets

c. twisted setting

- remarks on last calculation:
- (i) $\langle \pi, s \rangle$ is the absolute version of $\Delta_{\|}$ available in this setting in general setting, there is a central obstruction to defining $\omega$ in $H^1$ which is handled by going to relative version using a trick from the original definition of geometric factors in [L-S] [trick works for any pair $T, T'$ of maximal tori over local field $F$ ...]

- (ii) it is easy to extend this type of calculation (for discrete series) to the twisted setting using fundamental splittings (Weyl chambers $\rightsquigarrow$ fnd. splittings):
Structure on packets

c. twisted setting

- remarks on last calculation:
  - (i) $\langle \pi, s \rangle$ is the absolute version of $\Delta_{III}$ available in this setting. In general setting, there is a central obstruction to defining $\omega$ in $H^1$ which is handled by going to relative version using a trick from the original definition of geometric factors in [L-S] [trick works for any pair $T, T'$ of maximal tori over local field $F$ ...] nonabelian variant for general elliptic $u$-regular case

- (ii) it is easy to extend this type of calculation (for discrete series) to the twisted setting using fundamental splittings (Weyl chambers $\rightsquigarrow$ fnd. splittings):

- assume $\theta$ preserves fnd. splitting $spl_f$; may assume inner twist $\eta$ transports $spl_f$ to fnd. splitting $spl_f^*$ of $G^*$ preserved by $\theta^*$, $spl_f^*$ provides toral data to transport objects from $G^\vee$...
\[ \Pi = (\theta, \varpi) \text{-stable packet of discrete series} \]

find splitting \( spl_{\pi} \) for \( \pi \) in twist-packet \( \Pi^{\theta, \varpi} \) is preserved by \( \theta \)

up to inner automorphism \( \eta \) transports \( spl_{\pi} \) to \( spl_f^* \)

make Galois cocycle in this setting (relative in general)
Structure on packets

cc. twisted setting (cont.)

- $\Pi = (\theta, \varpi)$-stable packet of discrete series
  find. splitting $spl_\pi$ for $\pi$ in twist-packet $\Pi^{\theta,\varpi}$ is preserved by $\theta$
  up to inner automorphism $\eta$ transports $spl_\pi$ to $spl_f^*$
  make Galois cocycle in this setting (relative in general)

- cocycle almost takes values in $\theta^*$-invariants; instead,
  satisfies hypercocycle condition, so back to setting of
  Kottwitz-Shelstad for geometric transfer factors
• $$\Pi = (\theta, \omega)$$-stable packet of discrete series
  find. splitting $$spl_\pi$$ for $$\pi$$ in twist-packet $$\Pi^{\theta, \omega}$$ is preserved by $$\theta$$
  up to inner automorphism $$\eta$$ transports $$spl_\pi$$ to $$spl^*_f$$
  make Galois cocycle in this setting (relative in general)

• cocycle almost takes values in $$\theta^*$$-invariants; instead,
  satisfies hypercocycle condition, so back to setting of
  Kottwitz-Shelstad for geometric transfer factors

• compatibility statement: introduce twisted version of $$S$$,
Structure on packets
cc. twisted setting (cont.)

- $\Pi = (\theta, \omega)$-stable packet of discrete series
  - find splitting $\text{spl}_\pi$ for $\pi$ in twist-packet $\Pi^{\theta, \omega}$ is preserved by $\theta$
  - up to inner automorphism $\eta$ transports $\text{spl}_\pi$ to $\text{spl}_f^*$
  - make Galois cocycle in this setting (relative in general)
- cocycle almost takes values in $\theta^*$-invariants; instead, satisfies hypercocycle condition, so back to setting of Kottwitz-Shelstad for geometric transfer factors
- compatibility statement: introduce twisted version of $S$,
- work in $G^\vee \rtimes \langle \theta^\vee \rangle$ ...
Structure on packets
cc. twisted setting (cont.)

- \( \Pi = (\theta, \wp) \)-stable packet of discrete series
  - Find splitting \( spl_{\pi} \) for \( \pi \) in twist-packet \( \Pi^{\theta, \wp} \) is preserved by \( \theta \) up to inner automorphism \( \eta \) transports \( spl_{\pi} \) to \( spl_{f}^* \)
  - Make Galois cocycle in this setting (relative in general)

- Cocycle almost takes values in \( \theta^* \)-invariants; instead, satisfies hypercocycle condition, so back to setting of Kottwitz-Shelstad for geometric transfer factors

- Compatibility statement: introduce twisted version of \( S \),
  - Work in \( G^\vee \rtimes \langle \theta^\vee \rangle \) ...

- For nontrivial twisting character \( \wp \), analysis exploits map on endo data: \( \epsilon_z \rightarrow (\epsilon_z)_{ad} \) dual to \( G_{sc} \rightarrow G \)