THIN MATRIX GROUPS
AND THE MONODROMY
OF THE HYPERGEOMETRIC EQUATION.

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\[ \Gamma \leq \text{GL}_n(\mathbb{Z}) \]

\[ G = \text{Zarishii closure of } \Gamma \text{ if } \Gamma \text{ is algebraic} \]

so \[ \Gamma \leq G(\mathbb{Z}) \]

We say \( \Gamma \) is arithmetic if it is finite index in \( G(\mathbb{Z}) \) and thin if not.

Many diophantine problems, standard and more exotic are connected with orbits of such a \( \Gamma \): Fix \( \mathbf{u} \in \mathbb{Z}^n \),

\[ \mathcal{O} := \Gamma \cdot \mathbf{u} \subset \mathbb{Z}^n \]
Examples of problems:

(i) If $f \in \mathbb{Z}[x_1, \ldots, x_n]$ what values does $f$ assume on $\mathcal{O}$? Is there a local to global principle?

(ii) Can we find 'many' $x$'s in $\mathcal{O}$ at which $f(x)$ is prime or at least has few prime factors? ("Affine Sieve").

- In the case $\Pi$ is arithmetic these are classical (and can be very difficult) problems.
- In the case that $\Pi$ is thin the problem is much more challenging but we now have the rudiments of a theory.
A key ingredient is a weak form of the Ramanujan Conjectures for such thin $\Gamma$'s. These are given in terms of properties of the corresponding congruence graphs.

Fix generators $s_1, s_2, \ldots, s_t$ of $\Gamma$

$S = \{ s_1, s_1^{-1}, \ldots, s_t, s_t^{-1} \}$

$|S| = 2t$.

For $g \geq 1$

$\Gamma(g) \rightarrow \Gamma \rightarrow \text{reduction mod } g \rightarrow \text{GL}_n(\mathbb{Z}/g\mathbb{Z})$

$(\Gamma/\Gamma(g), S)$ finite Cayley graphs

$\exists \gamma(g) \sim s \gamma(g) \in S$.

Do these $|S|$ regular graphs form an expander family as $g \to \infty$?
Thanks to the work of many people [5-Xue], [Gamburd], [HeeLfgott], [Bourgain-Gamburd], [Bourgain-Gamburd-S], [Pyber-StabO], [Brvillard-Green-Tao], [Varju] we have

**Fundamental Expansion Theorem (Salehi-Varju)**

\((\pi/\pi(q), S)\) is an expander family iff \(G^0\) the identity component of \(G:=\text{Zcl}(\pi)\), is perfect (ie. \([G^0 : G^0 ] = G^0\)).

**Application: Affine Sieve**

If \(f \in \mathbb{Z}[x_1, \ldots, x_n]\) and \(O = \pi_1\)

we say that \((O, f)\) saturates if there is an \(T < \infty\) such that

\[\{x \in O : f(x) \text{ has at most } T \text{-prime factors} \}\]

is Zariski dense in \(\text{Zel}(O)\).

- The minimal such \(T\) is the saturation number \(T_0(O, f)\).
Example: (1) \( \Theta = \mathbb{Z} \), \( f(x) = x(x+2) \)
\( r_0(\Theta, f) = 2 \) \( \iff \) twin prime conjecture.

And

(2) Theorem Y. Zhang (yesterday).
\( r_0(\Theta, x(x+k)) = 2 \) for at least one even \( k \) less than \( 7 \cdot 10^7 \).

Fundamental Saturation Theorem - Affine Sieve
Salehi-S (2013):
\( \Gamma, f \) as above, \( \Theta = \Gamma v < \mathbb{Z}^n \).
If \( G = \text{Zcl}(\Gamma) \) is Levi semi-simple (i.e., \( \text{rad} G \) contains no torus) then
\( r_0(\Theta, f) < \infty \).

Heuristic arguments show that the condition on the radical of \( G \) is probably necessary for saturation.
For examples of local to global principles for Apollonian packings see the recent BAMS papers of Fuchs and Kontorovich.

**Ubiquity of Thin Groups?**

- There is no decision procedure to tell whether a given \( A_1, A_2, \ldots, A_l \) in \( SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \) generate a thin group or not (Mihalova 1959). In practice if \( \Gamma \) is in fact a congruence subgroup of \( G(\mathbb{Z}) \), and is given in terms of generators, then one can verify this by producing \( g \) generators of the congruence subgroup. However if \( \Gamma \) is thin — how do we certify this?

- For a true group theorist thin is the rule! Given \( A, B \in SL_n(\mathbb{Z}) \) chosen at random (say \( ||A||, ||B|| \) \( \times \) uniform measure) then with prob tending to one, \( \Gamma = \langle A, B \rangle \) has \( G = 2\sigma(\Gamma) = SL_n \), \( \Gamma \) is free and thin (Aoun, Fuchs).
HYPERBOLIC REFLECTION GROUPS (VINBERG):

\( f(x_1, x_2, \ldots, x_n) \) a rational quadratic form of signature \((n-1,1)\) \((n \geq 5)\).

\( G = O_f, \ G(Z) \) arithmetically.

Let \( R_f(Z) \) be the (normal) subgroup of \( G(Z) \) generated by all \( \beta \in G(Z) \) which induce hyperbolic reflections on \( H^{n-1} \),

Then except for finitely many special \( f \)'s

\[ |O_f(Z)/R_f(Z)| = \infty. \]

MONODROMY GROUPS: A natural geometric source of finitely generated subgroups of \( GL_n(Z) \) is the monodromy representation in cohomology of a family of algebraic varieties, variations of Hodge structures, monodromy of linear differential equations, ..
· The fundamental question as to whether in the case of variation of Hodge structures the monodromy $\pi$ is arithmetic was posed in 1973 by Griffiths / Schmid.

· They show that if the period map from the parameter space $\mathcal{S}$ to the period domain $D$ is open then $\pi$ is arithmetic.

McMullen (2012) considers cyclic covers of $\mathbb{P}^1$:

$C_d: y^d = (x-a_1)(x-a_2) \ldots (x-a_{n+1})$

The fundamental group of the parameter space of $C_d$'s is the (pure) braid group.

· Answering a question of McMullen Venkataramana (2013) shows that if $M \geq 2d$, the monodromy group in $\text{GL}(H_1(C)[\mathbb{Z}])$, C a base curve, is arithmetic!

· If $n=3$ and $d=18$, McMullen shows that the monodromy is thin using a relation to non-arithmetic lattices of Deligne--Mostow.
One parameter hypergeometric $\nu F_{n-1}$:

$x, \beta \in \mathbb{Q}^n$, $0 \leq \alpha_j < 1$, $0 \leq \beta_k < 1$

(*) $D u = 0$, $\Theta = \frac{\partial}{\partial z}$

$D = (\Theta + \beta_1 - 1) \cdots (\Theta + \beta_{n-1} - 2) (\Theta + \alpha_1) \cdots (\Theta + \alpha_n)$

Solutions are

$z^{-\beta} \nu F_{n-1} \left( 1 + \alpha_1 - \beta_1, \ldots, 1 - \alpha_n - \beta_n; 1 + \beta_1 - \beta_1, \ldots, 1 + \beta_{n-1} - \beta_{n-1} \mid z \right)$

where $\nu$ means omit $1 + \beta_i - \beta_i$ and

$\nu F_{n-1} \left( \alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n \mid z \right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_n)_k}{(\beta_1)_k \cdots (\beta_n)_k} \frac{z^k}{k!}$

(*) is singular at $0, 1, \infty$ and the monodromy group $H(x, \beta)$ is gotten by analytic continuation along paths in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ of a basis of solutions.

We restrict to $\alpha, \beta$ s.t. $H(x, \beta)$ up to conjugation is $\text{vi GL}_n(\mathbb{Z})$. 
Beukers-Heckman compute

\[ G = \text{ZcL} \left( H(\alpha, \beta) \right) \]

explicitly in terms of \( \alpha, \beta \).

In this self-dual setting it is one of

(i) Finite
(ii) On
(iii) \( \text{Sp}_n \) (only occurs if \( n \) is even)

Venkataramana (2012): \( n \geq 2 \) even

\[ \alpha = \left( \frac{1}{2} + \frac{1}{n+1}, \frac{1}{2} + \frac{2}{n+1}, \ldots, \frac{1}{2} + \frac{n}{n+1} \right) \]

\[ \beta = \left( 0, \frac{1}{2} + \frac{1}{n}, \frac{1}{2} + \frac{2}{n}, \ldots, \frac{1}{2} + \frac{n-1}{n} \right) \]

\[ G(\alpha, \beta) = \text{Sp}_n \text{ and } H(\alpha, \beta) \text{ is arithmetic!} \]

There are 112 \( (\alpha, \beta) \)'s giving

\[ G(\alpha, \beta) = \text{Sp}_4, \text{ all come from variations of integral Hodge structures.} \]

(DORAN-MORGAN)
Of these more than half are arithmetic (Singh-Venkataramanan 2012).

14 of these correspond to Calabi-Yau families of 3-folds.

eq: \((0,0,0,0), \left(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right)\)

\[
\text{Part of Dwork family, Candelas et al mirror symmetry family.}
\]

Brav-Thomas (2012) show that this example is thin!

They show that the generators \(q \in \mathcal{P}(\mathbb{P}^3, 0,1,003)\) A and C, about 0 and 1, play generalized ping-pong on some complicated polyhedral sets in \(\mathbb{P}^3\).

Of the 14 Calabi-Yau's 7 are thin and 3 arithmetic.
HYPERBOLIC HYPERGEOMETRIC (Fuchs-Meiri 2013)

- \((\alpha, \beta)\) is hhm if \(G(\alpha, \beta)\) is orthogonal and of signature \((n-1,1)\).
- \(n\) must be odd.

**Theorem 1 (F-M-S)**

With the exception of an explicit (long) list of finitely many \((\alpha, \beta)\) (all with \(n \leq 9\)), all hhm's come in 7 parametric families.

For the hhm we give a robust obstruction for \(H(\alpha, \beta)\) to be arithmetic — that is for \(H(\alpha, \beta)\) to be thin.
A rational quadratic form $f(x) = (x, x)$ is integral on the lattice $(m-1, 1)$.

$\langle x, x \rangle = -2 \quad \langle x, x \rangle = 0 \quad \langle x, x \rangle > 0$

$\langle x, x \rangle < 0$

If $(v, u) \neq 0$ then the linear reflection

$$\tau_v(y) = y - \frac{2(u, y)}{u, u} u$$

is in $O(u)$ if $u, u = \pm 1$.
If \((v, v) > 0\) then \(T_v\) induces a hyperbolic reflection on \(H^{n-1}\).

If \((v, v) < 0\) then \(T_v \in O_f\) induces a **Cartan** involution on \(H^{n-1}\).

Key observation 1: \((h, h_w)\)

\[H(\alpha, \beta) = \langle A, B \rangle\]

local monodromy \(A\) about 0 \(B\) about \(\infty\)

Then \(C = A^{-1}B\) is a **CARTAN** involution!

Up to commensurability \(H(\alpha, \beta)\) is generated by Cartan involutions.
Let
\[ R_2(L) = \sum_{u \in L} : (u, u) = 2 \mathbb{Z} \]
be the root vectors giving hyperbolic reflections.

\[ R_{-2}(L) = \sum_{u \in L} : (u, u) = -2 \mathbb{Z} \]
the root vectors giving Cartan involutions.

According to Vinberg / Nikulin, except for special \( f \)'s, \( |O(L)/R_2(L)| = \infty \).

Let \( \Delta \subset R_{-2}(L) \) we give a condition under which
\[ \langle T_u : u \in \Delta \rangle \]
has finite image in \( O(L)/R_2(L) \).
The points of $X(L)$ are the Cartan roots $R_{-2}(L)$ joined by $u$ to $w$ if $(u, w) = -3$.

( minimal distance these can be ! )

Lemma: \[ \text{If } \Delta \text{ is in a connected component of } X(L) \]

Then \[ \langle T_u : u \in \Delta \rangle \text{ has finite image in } O(L)/R_2(L). \]

This gives the obstruction to being arithmetic. Using it we have

\[ \text{THEOREM: } n \text{ odd.} \]

\[ \alpha = (0, \frac{1}{n+1}, \frac{2}{n+1}, \frac{n-1}{2(n+1)}, \frac{n+3}{2(n+1)}, \ldots, \frac{n}{n+1}), \beta = \left( \frac{1}{2}, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n} \right) \]

\[ \alpha = (\frac{1}{2}, \frac{1}{2n-2}, \frac{3}{2n-2}, \ldots, \frac{2n-3}{2n-2}), \beta = (0, 0, \frac{1}{n-2}, \frac{2}{n-2}, \ldots, \frac{n-1}{n-2}) \]

are hyperbolic hypergeometrics and are arithmetic if $n = 3$ and thus if $n \geq 5$. 