Local perturbations perturb locally

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joint work with S. Michalakis, B. Nachtergaele and R. Sims

Overview

The general interest: ground state subspaces of families of Hamiltonians

Today:

- A mapping between gapped spectra: the spectral flow
- Quantum lattice systems and the Lieb-Robinson bound
- Small commutators and locality

... and bringing everything together,

- Local perturbations perturb ground states only locally

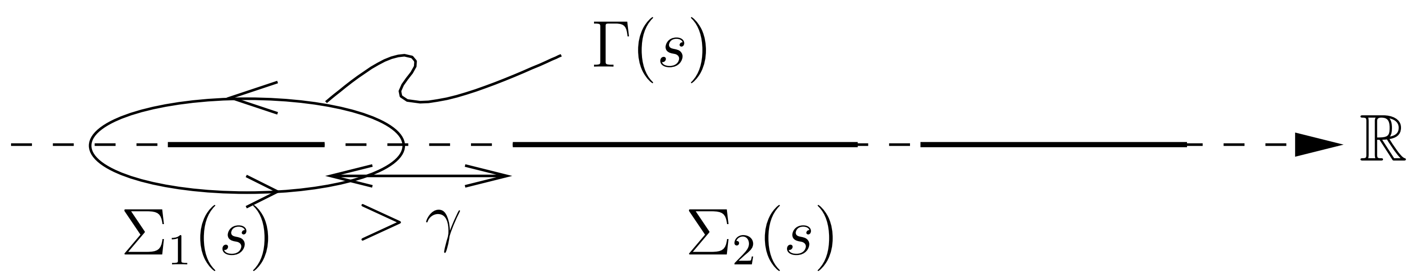
Parameter dependent Hamiltonians

We consider a smooth family of operators $\{H(s)\ :\ s \in [0,1]\}$ on a fixed Hilbert space $\mathcal{H}$ such that

- $H(s) = H(s)^*$ densely defined (can be unbounded)
- $H'(s)$ is uniformly bounded for $s \in [0,1]$
- Spectrum $\Sigma(s)$ has a uniform gap γ: $\Sigma(s) = \Sigma_1(s) \cup \Sigma_2(s)$, with

$$\Sigma_1(s) \cap \Sigma_2(s) = \emptyset\,, \qquad d(\Sigma_1(s), \Sigma_2(s)) > \gamma$$

for all s.



Spectral projections

Let $P(s)$ be the spectral projection on $\Sigma_1(s)$,

$$P(s) = -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma(s)} R(z,s) dz\,, \quad \text{where} \quad R(z,s) = (H(s)-z)^{-1}$$

Then,

$$P'(s) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma(s)} R(z,s)H'(s)R(z,s) dz\,.$$

Use $PP'P = 0$ and the spectral decomposition of $R(z,s)$:

$$\begin{aligned} P'(s) &= P(s)P'(s)(1-P(s)) + (1-P(s))P'(s)P(s) \\ &= -\int_{I(s)} d\mu \int_{\mathbb{R}/I(s)} d\lambda\, \frac{1}{\lambda-\mu} \big(dE_\mu(s)H'(s)dE_\lambda(s) + dE_\lambda(s)H'(s)dE_\mu(s) \big) \end{aligned}$$

with $|\lambda - \mu| > \gamma$ by assumption.

A weight function

Suppose that $w_\gamma \in L^1(\mathbb{R})$ is a real function such that $\int w_\gamma(t)dt = 1$. Observe then

$$\mathrm{i}\int dt\, w_\gamma(t) \int_0^t du\, \mathrm{e}^{\pm \mathrm{i}u(\lambda-\mu)} = \pm \int dt\, w_\gamma(t) \frac{1}{\lambda-\mu}\big(\mathrm{e}^{\pm \mathrm{i}t(\lambda-\mu)} - 1\big) = \mp \frac{1}{\lambda-\mu},$$

whenever the Fourier transform is compactly supported, namely

$$\mathrm{supp}(\widehat{w}_\gamma) \subset [-\gamma, \gamma].$$

Note: such functions exist, e.g.

$$w_\gamma(t) = \frac{1}{Z_\gamma} \cdot \prod_{n=1}^{\infty} \Big(\frac{\sin a_n t}{a_n t}\Big)^2, \qquad \text{with} \qquad \sum_{n=1}^{\infty} a_n = \gamma/2$$

The spectral flow

Conclude:

$$P'(s) = \mathrm{i}\big((1-P)DP - PD(1-P)\big) = \mathrm{i}[D(s), P(s)]$$

with

$$D(s) = \int_{-\infty}^{\infty} dt\, w_\gamma(t) \int_0^t du\, \mathrm{e}^{\mathrm{i}uH(s)} H'(s) \mathrm{e}^{-\mathrm{i}uH(s)} = D(s)^* \,.$$

In other words, $P(s)$ is obtained from $P(0)$ by a unitary evolution $U(s)$ with explicitly known generator $D(s)$, namely

$$-\mathrm{i}\frac{d}{ds}U(s) = D(s)U(s)$$
$$U(0) = 1$$

Useful tool: The spectral flow

Theorem. [Hastings ’04, B-Michalakis-Nachtergaele-Sims ’11]
The spectral projections $P(s)$ are unitary conjugates of each other

$$P(s) = U(s)P(0)U(s)^*$$

where $U(s)$ is the unitary group generated by

$$D(s) = \int_{-\infty}^{\infty} W_\gamma(t)\mathrm{e}^{\mathrm{i}tH(s)}H'(s)\mathrm{e}^{-\mathrm{i}tH(s)}dt\,,$$

where $W_\gamma' = w_\gamma$.

- The new form of $D(s)$ follows by integration by parts.
- With particular choice made above: Almost exponential decay

$$|W_\gamma(t)| \lesssim \exp\left(-\mathrm{const}\cdot\frac{\gamma t}{(\ln\gamma t)^2}\right)$$

Quantum spin models

- A countable collection of quantum systems, labelled by $x \in \Gamma$, with (finite dimensional) Hilbert spaces $\mathcal{H}_x$, not necessarily identical
- Typically: Γ is a lattice or a graph (equipped with distance $d(\cdot,\cdot)$) and $\mathcal{H}_x$ is the Hilbert space of states of a spin of magnitude S, $\mathcal{H}_x = \mathbb{C}^{2S+1}$
- The total Hilbert space $\mathcal{H}_\Gamma$ is the tensor product of the local state spaces,

$$\mathcal{H}_\Gamma = \bigotimes_{x\in\Gamma} \mathcal{H}_x$$

- The local algebra of observables for finite $\Lambda \subset \Gamma$ is

$$\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{L}(\mathcal{H}_x)$$

the set of bounded operators on the local Hilbert space $\mathcal{L}(\mathcal{H}_\Lambda)$

Quasi-local observables and Hamiltonians

- Natural identification, for $\Lambda_1 \subset \Lambda_2$, $B \in \mathcal{A}_{\Lambda_1} \Rightarrow B \otimes 1_{\Lambda_2 \setminus \Lambda_1} \in \mathcal{A}_{\Lambda_2}$. Hence, $\Lambda_1 \subset \Lambda_2 \Rightarrow \mathcal{A}_{\Lambda_1} \subset \mathcal{A}_{\Lambda_2}$
- The quasi-local algebra is the limit of finite dimensional matrix algebras

$$\mathcal{A}_\Gamma = \overline{\bigcup_{\Lambda \subset \Gamma} \mathcal{A}_\Lambda}$$

- For each finite system, we define a Hamiltonian $H_\Lambda \in \mathcal{A}_\Lambda$ which generates the Heisenberg automorphism $A \mapsto \tau_t^{H_\Lambda}(A)$ on $\mathcal{A}_\Lambda$.
- Local interactions: For $\Phi(X) \in \mathcal{A}_X$

$$H_\Lambda = \sum_{X \subset \Lambda} \Phi(X)$$

with suitable decay of $\Phi(X)$ on the size of X

Lieb-Robinson bounds

Consider local observables $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$, supported away from each other, $d(X, Y) = d > 0$. In particular

$$[\tau_0^{H_\Lambda}(A), B] = [A, B] = 0\,.$$

There is a constant $C(A, B)$, and two constants μ and v depending on the rate of decay of Φ, such that the Lieb-Robinson estimate holds

$$\left\|[\tau_t^{H_\Lambda}(A), B]\right\| \leq C(A, B)\mathrm{e}^{-\mu(d - v|t|)}$$

For times $t \leq d/v$, $\tau_t^{H_\Lambda}(A)$ almost commutes with B
One classical application: prove the existence of the thermodynamic limit of the dynamics $\tau_t^{H_\Lambda}(\cdot)$ as Λ tends to Γ

Almost commuting operators

Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. If $A \in \mathcal{L}(\mathcal{H}_1)$ and $B \in \mathcal{L}(\mathcal{H}_2)$, then $[A, B] = 0$.
Also: if $A \in \mathcal{L}(\mathcal{H})$ and

$$[A, B] = 0 \quad \forall B \in \mathcal{L}(\mathcal{H}_2) \qquad \Longrightarrow \qquad A \in \mathcal{L}(\mathcal{H}_1)$$

Lemma. *Suppose that* $\epsilon \geq 0$ *and* $A \in \mathcal{L}(\mathcal{H})$ *are such that*

$$\|[A, 1 \otimes B]\| \leq \epsilon \|B\| \text{ for all } B \in \mathcal{L}(\mathcal{H}_2).$$

Then, there exists $\Pi(A) \in \mathcal{L}(\mathcal{H}_1)$ *such that*

$$\|\Pi(A) \otimes 1 - A\| \leq 2\epsilon.$$

In other words, almost commutation implies almost localization
$\rightsquigarrow$ interpretation of LR bounds as a propagation estimate

Conditional expectation as partial trace

For a finite dimensional Hilbert space $\dim(\mathcal{H}_2) < \infty$, we can choose

$$\Pi(A) = \frac{1}{\dim(\mathcal{H}_2)} \cdot \mathrm{Tr}_{\mathcal{H}_2} A \in \mathcal{L}(\mathcal{H}_1)$$

Indeed,

$$\Pi(A) \otimes 1 = \int_{\mathcal{U}(\mathcal{H}_2)} d\mu(U)\, (1 \otimes U^*) A (1 \otimes U)$$

where μ is the Haar measure on the unitary group $\mathcal{U}(\mathcal{H}_2)$, of $\mathcal{H}_2$.
By assumption,

$$\|\Pi(A) \otimes 1 - A\| \leq \int_{\mathcal{U}(\mathcal{H}_2)} d\mu(U)\, \|(1 \otimes U^*)[A, (1 \otimes U)]\| \leq \epsilon\,.$$

In the infinite dimensional $\mathcal{H}_2$, $\Pi := 1 \otimes \rho$, for a normal state ρ

On the general case

Assume that $\mathcal{H}_2$ is infinite dimensional. For any finite dimensional projection $P \in \mathcal{L}(\mathcal{H}_2)$, let $\mathcal{U}(P)$ be the compact group of unitary operators of the form

$$\mathcal{U}(P) \ni U_P = (1 - P) + PUP \,,$$

and let $\Pi_P(A) = \int_{\mathcal{U}(P)} d\mu_P(U_P)\,(1 \otimes U_P^*)A(1 \otimes U_P)$. Then again, $\|\Pi_P(A) \otimes 1 - A\| \leq \epsilon\|A\|$.

To conclude: Choose an increasing net P_λ converging to 1 on $\mathcal{H}_2$. Then $(\Pi_{P_\lambda})_\lambda$ is bounded, therefore weakly-* convergent, to Π_∞.

Problem: $A \mapsto \Pi_\infty(A)$ not necessarily continuous. Therefore, choose another map $\Pi = 1 \otimes \rho$: Gain continuity w.r.t. the weak operator topology (ρ is normal) but loose an ϵ, see [Nachtergaele, Scholz, Werner '11]

$$\|\Pi(A) \otimes 1 - A\| \leq 2\epsilon \,.$$

A local perturbation

As a first example: We use the spectral flow, the LR bound and the conditional expectation to analyze ground states.

- We now consider a finite lattice system, with possibly $\dim(\mathcal{H}_x)$ infinite, satisfying a LR bound
- Let $H(0)$ and $H(1)$ be two Hamiltonians with gapped ground states $\Psi(0)$ and $\Psi(1)$ and such that

$$H(1) = H(0) + \Phi(X)\,, \qquad \Phi(X) \in \mathcal{A}_X\,.$$

  i.e. $H(1)$ is a local perturbation of $H(0)$.

- Assume that there exists a path $H(s)$ of uniformly gapped Hamiltonians $H(s)$ between them, with $H'(s) = \Phi'(X, s) \in \mathcal{A}_X$ for all s
- Question: How different are $\Psi(1)$ and $\Psi(0)$, away from X?
  $\rightsquigarrow$ Use the spectral flow to compare the states

Local perturbation and the spectral flow

Recall:

$$D(s) = \int_{-\infty}^{\infty} W_\gamma(t)\tau_t^{H(s)}\big(H'(s)\big)dt$$

In general, $D(s)$ is not local, even for strictly local $H'(s)$. But:

- Lieb-Robinson bound: For t not too large, $t \leq T$,

$$\left\|\tau_t^{H(s)}\big(\Phi'(X,s)\big) - \Pi_{X_R}\left(\tau_t^{H(s)}\big(\Phi'(X,s)\big)\right)\right\| \leq C(A,B)\mathrm{e}^{-\mu(R-v|t|)}$$

  where Π_{X_R} is the conditional expectation onto an R-fattening of X.

- Decay of W_γ: For $t > T$,

$$W_\gamma(t) \lesssim \exp\left(-\mathrm{const}\cdot\frac{\gamma t}{(\ln\gamma t)^2}\right)$$

and optimize on T.

Local generator

In fact, $D(s)$ can be approximated by a local version $D_R(s)$,

$$D_R(s) = \int_{-\infty}^{\infty} dt\, W_\gamma(t) \Pi_{X_R}(\mathrm{e}^{\mathrm{i}tH(s)} H'(s) \mathrm{e}^{-\mathrm{i}tH(s)}) \in \mathcal{A}_{X_R}$$

namely, $\|D_R(s) - D(s)\|$ decays almost exponentially in R.
Using this approximation, we prove

Theorem. [B-Michalakis-Nachtergaele-Sims '11]
For $R > 0$, there exists a unitary operator V_R, supported in X_R, and constants κ and C such that

$$\|P_{\psi(1)} - V_R P_{\psi(0)} V_R^*\| \leq \kappa \exp\left(-\mathrm{const} \cdot \frac{\gamma R/(2v)}{\ln^2(\gamma R/(2v))}\right)$$

Proof: V_R is generated by D_R.

Expectation values

Let $A \in \mathcal{A}_{\Lambda \setminus X_R}$ be an observable supported away from the perturbation, whence $[A, V_R(1)] = 0$. We immediately obtain

$$\begin{aligned} |\langle \psi(1), A\psi(1)\rangle - \langle \psi(0), A\psi(0)\rangle| &= |\langle \psi(0), U(1)^*[A, U(1)]\psi(0)\rangle| \\ &= |\langle \psi(0), U(1)^*[A, U(1) - V_R(1)]\psi(0)\rangle| \\ &\leq 2\|A\|\|U(1) - V_R(1)\| \\ &\lesssim \|A\| \exp\left(-\text{const} \cdot \frac{\gamma R/(2v)}{\ln^2(\gamma R/(2v))}\right) \end{aligned}$$

$\rightsquigarrow$ Therefore, the ground state $\psi(1)$ is exponentially weakly perturbed away from the perturbation $\Phi(X, 1)$.

Other applications and developments

The spectral flow has already had many other applications, among them:

- 'Quantum phases' and automorphic equivalence of ground state subspaces in the thermodynamic limit (see talk by R. Sims)
- Stability of topological phases (see talk by S. Michalakis)
- Quantum Hall effect (ask S. Michalakis)

... and further,

- Classification of gapped phases (see talks by B. Nachtergaele, N. Schuch)

More generally (renew grant...)

- Quantum phase transitions
- Quantum quenches