Review Notes – Miscellaneous Precalculus Topics

Important Information:

1. According to the most recent information from the Registrar, the Xa final exam will be held from 9:15 a.m. to 12:15 p.m. on Monday, January 13 in Science Center Lecture Hall D.

2. The test will include twelve problems (each with multiple parts).

3. You will have 3 hours to complete the test.

4. You may use your calculator and one page (8" by 11.5") of notes on the test.

5. I have chosen these problems because I think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the test will resemble these problems in any way whatsoever.

6. Remember: On exams, you will have to supply evidence for your conclusions, and explain why your answers are appropriate.

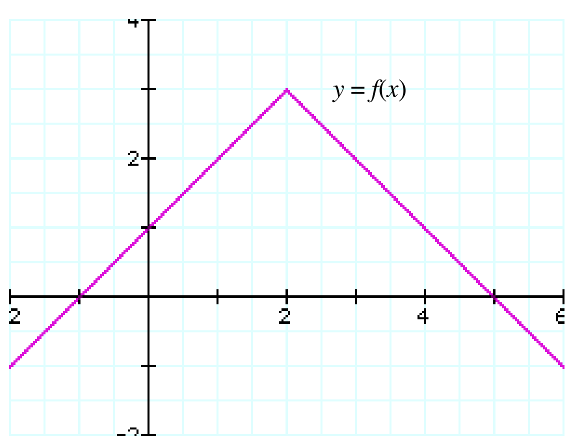
7. Good sources of help:

   - Section leaders' office hours (posted on Xa web site).
   - Math Question Center (during the reading period).
   - Course-wide review on Friday 1/10 from 4:00-6:00 p.m. in Science Center E and Sunday 1/12 from 3:00-5:00 p.m. in Science Center A.

1. Functions Defined in Pieces

Sometimes, it is much easier to describe a function algebraically using two or more equations than to try to find a single equation that will give the value of the function at each and every x-value.

For example, even a function with as simple a shape as the one shown below cannot easily be described using a single equation.



However the *y*-values of this function can be simply computed using the pair of formulas:

- $y = x + 1$ when $x \le 2$
- $y = 5 - x$ when $x > 2$.

The usual mathematical notation for specifying the collection of equations that are used to calculate the values of the function $f(x)$ is called piecewise function notation. Piecewise function notation includes both the formulas and the intervals of *x*-values for which each individual formula is used. The piecewise function notation for the function $f(x)$ whose graph is pictured above is:

$$f(x) = \begin{cases} x+1 & ,x \le 2 \\ 5-x & ,x > 2 \end{cases}$$

1.1 Example: Sketching the Graph of a Function Defined in Pieces

Sketch the graph of the function that is defined by:

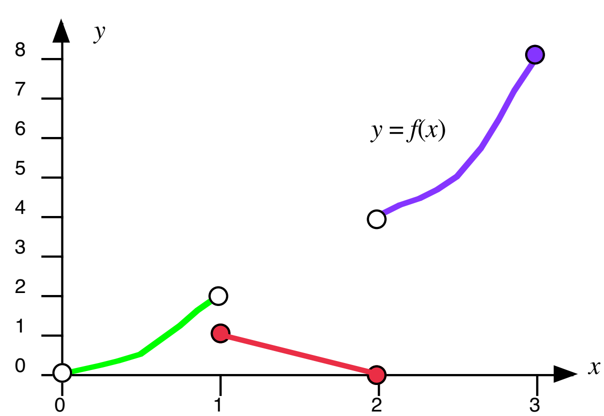
$$f(x) = \begin{cases} 2x^2 & ,0 < x < 1 \\ 2-x & ,1 \le x \le 2 \\ 2^x & ,2 < x \le 3 \end{cases}.$$

Solution

The first step in doing this will be to decode the meaning of the symbols, and decide which equation applies to which set of *x*-values.

- $y = 2x^2$ applies when $0 < x < 1$. As neither endpoint is included, the endpoints of this portion of the graph will be represented by open circles.
- $y = 2 - x$ applies when $1 \le x \le 2$. As both endpoints are included, the endpoints of this portion of the graph will be represented by filled in circles.
- $y = 2^x$ applies when $2 < x \le 3$. The left endpoint will be indicated with an open dot and the right endpoint will be indicated with a filled in dot.

The completed graph will resemble the following.



2. Inverses

First, let's look at some of the examples that you completed in Math Xa, and in doing do, review function notation.

The Uighurs and the Desert Mummies

In order to decide whether or not the Uighurs had lived in Western China for longer than the Chinese government said, we set up a function that used age as the input and gave the amount of carbon-14 present as the output.

INDEPENDENT VARIABLE	T	AGE IN YEARS
	↓	
FUNCTION	U	
	↓	
DEPENDENT VARIABLE	$U(T)$	AMOUNT OF C-14

But what we were really interested in doing was starting with an amount of carbon-14 and determining the age of one of the desert mummies.

AIDS orphans in South Africa

In order to decide how much trouble South Africa was in, we set up a function that took years as the input and gave the percentage of the population who were children orphaned by AIDS as the output.

INDEPENDENT VARIABLE	T	YEARS SINCE 1990
	↓	
FUNCTION	A	
	↓	
DEPENDENT VARIABLE	$A(T)$	PERCENTAGE OF POPULATION

But what we were really interested in doing was starting with a percentage (100%) and calculating the year when this would be achieved.

Biomedical data lab

In order to determine how long an elephant seal could remain submerged, we set up a function that took submergence time as the input and gave heart rate as the output.

INDEPENDENT VARIABLE	T	TIME SUBMERGED
	↓	
FUNCTION	H	
	↓	
DEPENDENT VARIABLE	$H(T)$	HEART RATE OF SEAL

But what we were really interested in doing was starting with a heart rate (zero) and calculating the time submerged that went with this.

2.1 The concept of the inverse

The common factor in each of the three examples is that we set up a function, but what we eventually needed to do was reverse the direction of the function. That is, instead of always being given x and then working out y, we were given y and asked to work back to find x.

The mathematical relationship that reverses the roles of the dependent and independent variable is called an **inverse relationship**. If a function uses x as its input and gives y as its output, then the **inverse relationship** takes y as the input and gives x as the output.

The inverse relationship for the Uighurs and the desert mummies

The inverse relationship would take the amount of carbon-14 as its input and give the age of the mummy as its output.

	T	AGE IN YEARS	$T = U^{-1}(c)$	
	↓		↑	
FUNCTION	U		U^{-1}	INVERSE
	↓		↑	
	$c = U(T)$	AMOUNT OF C-14	c	

South African AIDS orphans

The inverse relationship would take the percentage as its input and give the years since 1990 as its output.

	T	YEARS SINCE 1990	$T = A^{-1}(p)$	
	↓		↑	
FUNCTION	A		A^{-1}	INVERSE
	↓		↑	
	$p = A(T)$	PERCENTAGE	p	

Biomedical data lab

The inverse relationship would take the seal's heart rate as its input and give the time submerged as its output.

	T	TIME SUBMERGED	$T = H^{-1}(r)$	
	↓		↑	
FUNCTION	H		H^{-1}	INVERSE
	↓		↑	
	$r = H(T)$	HEART RATE	r	

In general, the link between a function and its inverse relationship is as follows.

INDEPENDENT VARIABLE	x	$x = f^{-1}(y)$	DEPENDENT VARIABLE
	↓	↑	
FUNCTION	f	f^{-1}	INVERSE
	↓	↑	
DEPENDENT VARIABLE	$y = f(x)$	y	INDEPENDENT VARIABLE

The inverse of a particular function, $y = f(x)$, can be written with the notation: $x = f^{-1}(y)$ where the "-1" indicates that it is the inverse and not the function that we talking about.

Since the inverse is basically the "reverse" of the function, using y-values as inputs and giving x-values as outputs, the range of the original function will be the **domain of the inverse**, and the domain of the original function will be the **range of the inverse**.

2.2 Is the inverse also a function?

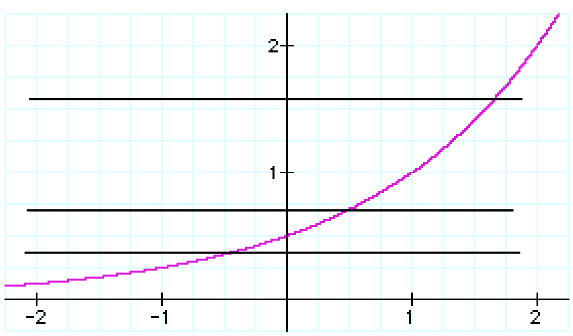
It depends a lot on the particular function $f(x)$ that you are talking about.

NOTE: If someone says that a function f is **invertible** (at least in Math Xa) then **what that means is that the inverse of *f* is also a function in its own right.**

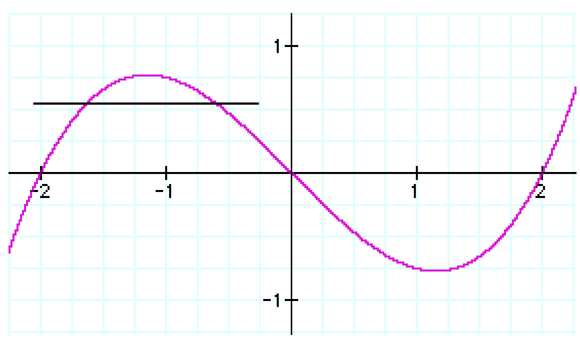
2.2.1 Checking to see if an inverse is a function in its own right

This is always easiest to do when the function is represented in a graphical way. The test is known as the **Horizontal Line Test**, and it is essentially the vertical line test for functions except with x and y switched. (Hence why it uses horizontal rather than vertical lines.)

1. Draw the graph of the function in the normal way with the independent variable graphed on the horizontal axis and the dependent variable graphed on the vertical axis.
2. If every horizontal line that you can draw cuts the graph in **one** place (or misses the graph altogether) then the inverse is a function in its own right.
3. If any horizontal lines cut the graph in **more than one place**, the inverse is **not** a function in its own right.



Horizontal lines only cut in one place each. The inverse is a function in its own right.



Some horizontal lines only cut in more than one place. The inverse is not a function in its own right.

2.3 Example: Finding Inverses of Functions Defined Numerically, Graphically and with Algebra

2.3.1 Numerical point of view

When a function is represented in a numerical format, it is usually as a table of values. The inverse will also be a table of values – just with the independent and dependent variables switched.

Function:

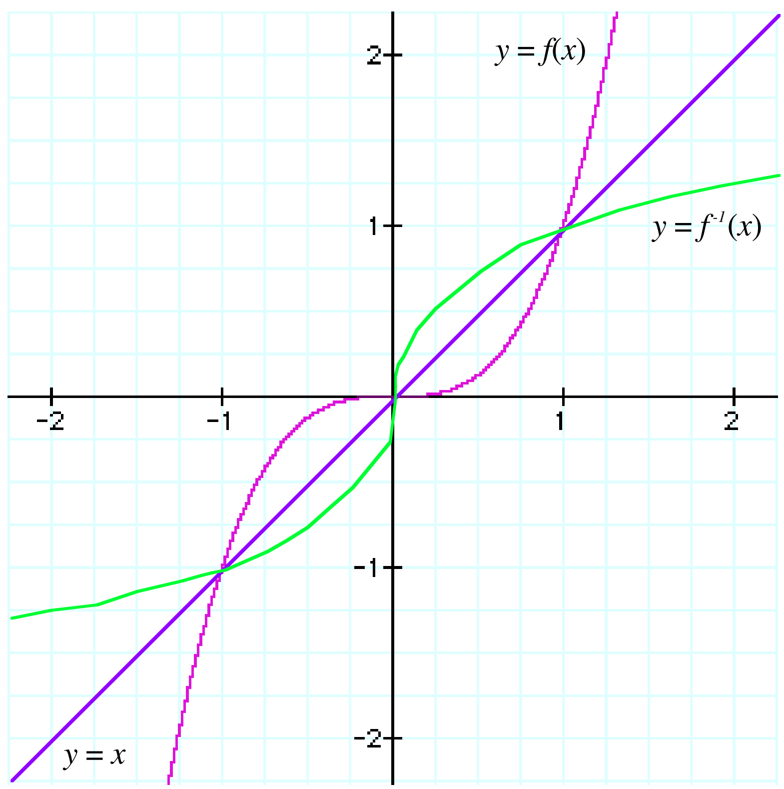
Independent variable (x)	0	1	2	3	4
Dependent variable (y)	9	11	13	15	17

Inverse:

Independent variable (y)	9	11	13	15	17
Dependent variable (x)	0	1	2	3	4

2.3.2 Graphical point of view

When a function is represented in a graphical format, the inverse will be the graph obtained by reflecting in the line $y = x$.



2.3.3 Algebraic point of view

When the function is represented in an algebraic format (with independent variable x and dependent variable y), an equation for the inverse can be obtained by **rearranging the function to make x the subject of the equation**.

For example, an equation for the inverse of:

$$y = f(x) = \frac{x}{x+1}$$

can be obtained by just re-arranging the equation to make x the subject.

$$y \cdot (x+1) = x$$

$$y \cdot x + y = x$$

$$y = x - x \cdot y$$

$$y = x \cdot (1 - y)$$

$$x = f^{-1}(y) = \frac{y}{1-y}.$$

2.4 Example: Not every inverse is a function in its own right

If we were given a y-value (say $y = +4$) that was generated by the function:

$$f(x) = x^2.$$

Could we unambiguously decide what x value had been used?

Solution

The big problem here is that =when you give the inverse a value (say 4) there could be more than one result of "undoing the function f" (e.g. -2 and +2).

So, if you are given a "y-coordinate" that lies on the graph of $y = f(x)$ there is no way to know for sure which "x-coordinate" it corresponds to.

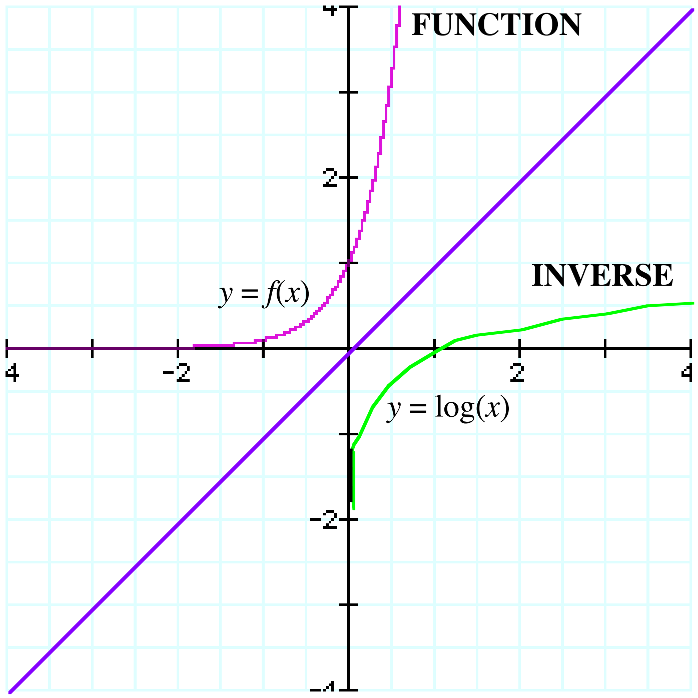
Therefore, the inverse of $f(x)$ is not an inverse in its own right.

3. Solving Equations Using Logarithms

The inverse of the function $f(x) = 10^x$ is one of the most useful functions in applications of mathematics. This inverse is used and referred to so often that it has its own name – the common logarithm (or just logarithm) function, and is denoted:

$$y = \log(x).$$

One point to observe (see following graph) is that the logarithm function is an increasing, concave down function. This provides you with another option that you can draw upon when you are examining a scatter plot and trying to decide what kind of function would do a decent job of representing the main trend in the data.



3.1 Properties of the Logarithm Function

The important properties of the logarithm function are (*A* and *B* are assumed to be positive numbers):

a. $\log\left(A^T\right) = T \cdot \log(A)$

b. $\log(A \cdot B) = \log(A) + \log(B)$

c. $\log\left(\frac{A}{B}\right) = \log(A) - \log(B)$

d. $\log\left(10^T\right) = T$

e. $10^{\log(A)} = A$

Property (a) (a.k.a. the super fun happy rule) is the most important for the purposes of solving exponential equations. The graph that we drew of the logarithm function also shows that logarithm has the following properties:

- The domain of the function $g(x) = \log(x)$ is the set $x > 0$.
- $\log(1) = 0$.
- If $0 < x < 1$ then $\log(x) < 0$. If $x > 1$ then $\log(x) > 0$.

3.2 Example: Using Logarithms to Solve an Exponential Equation

One of the major tools in modern archaeology is radiocarbon dating. Carbon-14 is a naturally occurring, radioactive isotope of carbon with a half-life of 5730 years. A fresh 100 g sample of organic matter will normally contain 0.0001 μg of carbon-14.

(a) Use the information provided to create a function that will give the amount (in micrograms, μg) that would be expected to remain in a 100 g sample of organic matter. The independent variable in your equation should be T, the age of the organic matter in years.

(b) Use the equation that you found in Question (a) to predict the amount of carbon-14 that would remain in a 100 g sample of organic matter that is 12,000 years old.

(c) The oldest human remains found in the Americas were discovered in 1975 in Lapa Vermelha, Brazil. The remains consisted of the skeleton of a woman in her 20's. The discovery was made Annete Emperaire, and the skeleton was nicknamed "Luzia." "Facial reconstruction" is a technique often used in police work. Clay is applied to a skull to create an approximation of the person's physical appearance when still alive. The facial reconstructions that have been made for Luzia are controversial and remarkable, because they suggest that Luzia may have more strongly resembled a person from Africa or Australia than the people living in South America today. Tests showed that a 100g sample from Luzia contained 0.0000249 μg of carbon Use logarithms to calculate the age of Luzia.

(d) The name "Kennewick man" refers to a skeleton that was discovered in 1996 near the Columbia River in Washington State. This skeleton was found by a group of college students who were vacationing in the area. When discovered, Kennewick man was thought to have been the victim of a recent homicide. Forensic investigation[1] quickly revealed that the skeleton had been buried beside the river for a considerable period of time. A facial reconstruction performed on the skull of Kennewick man suggested that his appearance may have been more typical of European/Caucasian people than Native American people. (One facial reconstruction of Kennewick man bore an uncanny resemblance to British actor Patrick Stewart, who played Captain Jean-Luc Picard on "Star Trek: The Next Generation.") These observations gave rise to a theory that Kennewick man may have been an early European settler who died at some time during the 1800's. Tests showed that a 100g sample from Kennewick man contained 0.0000327 μg of carbon-14. Use Logarithms to calculate the age of Kennewick man. Could Kennewick man have died during the 1800's?

Solution

(a) Let T = age of 100g sample be the independent variable and M = mass of carbon-14 remaining (in μg) be the dependent variable. The equation connecting these is:

$$M = 0.0001\cdot (1/2)^{T/5730} = 0.0001\cdot(0.9998790392)^{T}.$$

(b) $M(12000) = 0.00002354$ μg of carbon-14.

(c)

0.0000249	=	$0.0001\cdot(0.9998790392)^{T}$	
0.249	=	$(0.9998790392)^{T}$	(Divide by 0.0001)
log(0.249)	=	$T\cdot\log(0.9998790392)$	(Take logs)

[1] Examination of the skeleton revealed the presence of a stone spear point lodged in the bone of the pelvis. The investigators realized that in recent years, there had been very few reports of stab wounds to the pelvis involving stone-age weapons. They cited this as strong evidence to suggest that the skeleton was not the victim of a recent homicide.

log(0.249)/log(0.9998790392)	=	T	(Make T subject)
11493.13	=	T	(Evaluate on calculator)

So, Luzia is about 11,493 years old.

(d)

0.0000327	=	$0.0001\cdot(0.9998790392)^T$	
0.327	=	$(0.9998790392)^T$	(Divide by 0.0001)
log(0.327)	=	$T\cdot\log(0.9998790392)$	(Take logs)
log(0.327)/log(0.9998790392)	=	T	(Make T subject)
9240.41	=	T	(Evaluate on calculator)

So, Kennewick man is about 9,240 years old. It seems that Kennewick man was not a settler who died in the 1800's.

3.3 Example: Using Logarithms to Find the Intersection of Two Exponential Functions

In this example, we will use the rules of logarithms to find the point where a pair of exponential functions:

$p(x) = 3\cdot 5^x$ and $q(x) = 9\cdot 2^x$.

intersect. The two main logarithm rules that we will use will be:

- $\log\left(A^T\right) = T\cdot\log(A)$
- $\log\left(\frac{A}{B}\right) = \log(A) - \log(B)$

Solution

First, so that we have something to check our answer against, we are going to find the point of intersection using a graphic calculator.

1. Enter $p(x) = 3\cdot 5^x$ into your calculator as Y1 and $q(x) = 9\cdot 2^x$ into your calculator as Y2.
2. Set the size of your graphing window to: xmin=0, xmax=3, ymin=0, ymax=50.
3. Graph the two functions and use the INTERSECT feature to find the intersection point.

The x-coordinate of the intersection point is $x = 1.1989778$.

To find this intersection point using logarithms, we begin by setting the two equations to be equal to each other.

$$3\cdot 5^x = 9\cdot 2^x$$

It is usually a good idea to try to collect everything that involves an x on one side of the equation, and everything that doesn't involve x on the other side. This can be achieved by first dividing both sides by 3, and then by 2^x.

Dividing by 3: $$5^x = \frac{9\cdot 2^x}{3}$$

Dividing by 2^x: $$\frac{5^x}{2^x} = \frac{9}{3}$$

Next, we will apply logarithms to both sides of this equation and use the rule $\log\left(\frac{A}{B}\right) = \log(A) - \log(B)$ on the left side of the equation. (We could also use the rule on the right side of the equation, although that wouldn't achieve all that much.)

Apply logarithms: $$\log\left(\frac{5^x}{2^x}\right) = \log\left(\frac{9}{3}\right)$$

Use rule: $$\log(5^x) - \log(2^x) = \log\left(\frac{9}{3}\right)$$

Next, we will apply the rule $\log(A^T) = T \cdot \log(A)$ to both of the terms on the left side of the equation. This will finally release x from inside of the logarithms and allow us to solve the equation to find the numerical value of x.

$$x \cdot \log(5) - x \cdot \log(2) = \log\left(\frac{9}{3}\right)$$

Next, we will factor x out of each term on the left side of the equation,

$$x \cdot [\log(5) - \log(2)] = \log\left(\frac{9}{3}\right)$$

and then divide both sides of the equation by [log(5) − log(2)] to make x the subject.

$$x = \frac{\log\left(\frac{9}{3}\right)}{[\log(5) - \log(2)]}$$

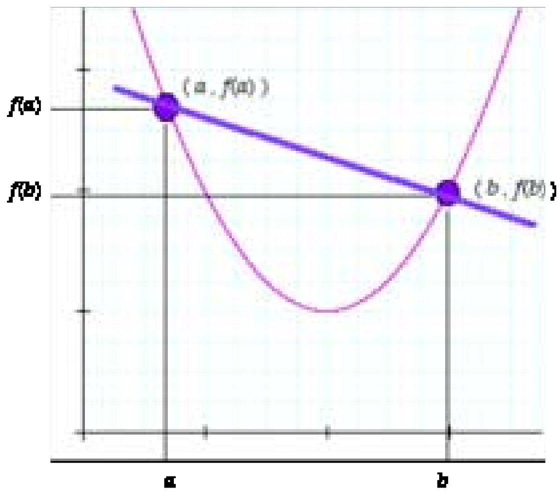
Evaluating this on a calculator gives $x = 1.1989778$ which is exactly the same as the answer we got when solving the equation using the INTERSECT capability of the calculator.

4. Rates and Concavity

4.1 The Average Rate of Change of a Function Over an Interval

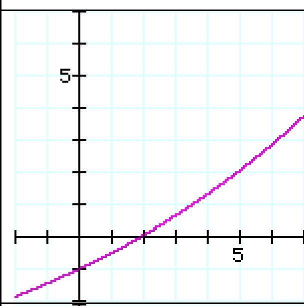
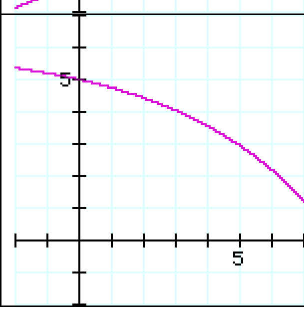
The slope of the line joining the points $(a, f(a))$ and $(b, f(b))$ is the change in y (the **rise**) over the change in x (the **run**). In an algebraic format this will be:

$$Slope = \frac{f(b) - f(a)}{b - a}.$$

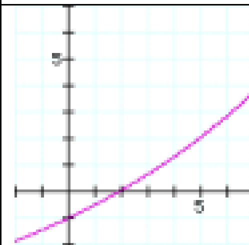
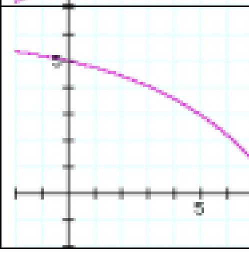


This slope equals the average rate of change of the function $f(x)$ over the interval between $x = a$ and $x = b$.

4.2 Rate of Change and Increasing/Decreasing of Original Function

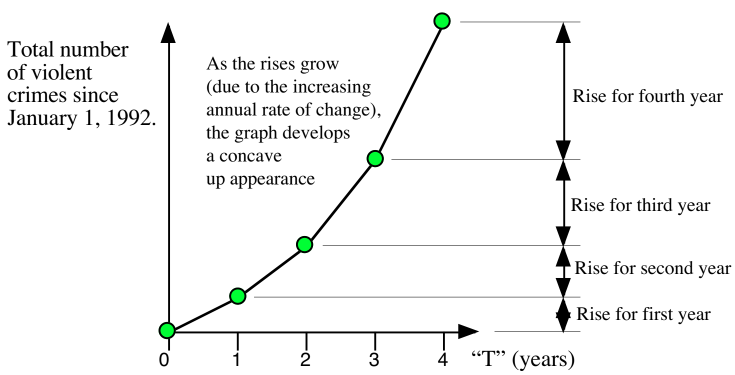
Rate of change	Behavior of function	Picture
Positive	Increasing	
Negative	Decreasing	

4.3 Rate of Change and Concavity

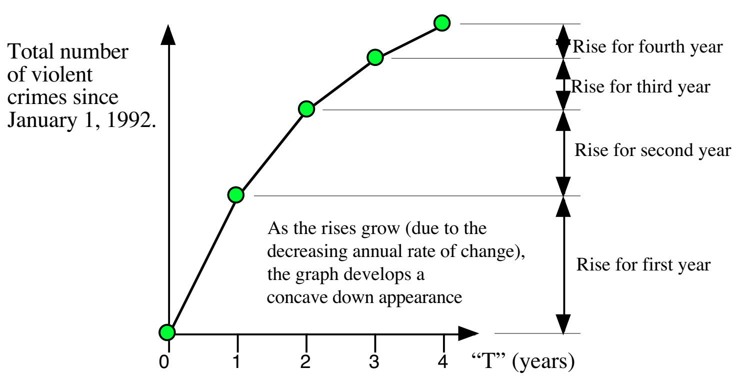
Rate of change	Behavior of function	Picture
Increasing	Concave up	
Decreasing	Concave down	

4.3.1 Understanding the Relationship Between Rate of Change and Concavity

Graph 1: Increasing rate of change leads to a concave up graph



Graph 2: Decreasing rate of change leads to a concave down graph

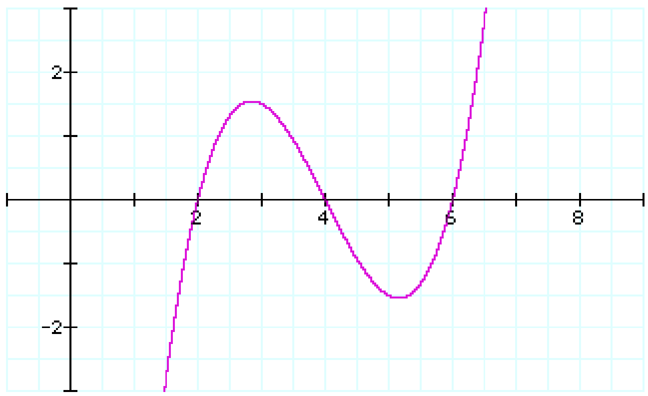


4.4 Example: Reconstructing an Original Function Graph from the Rate of Change

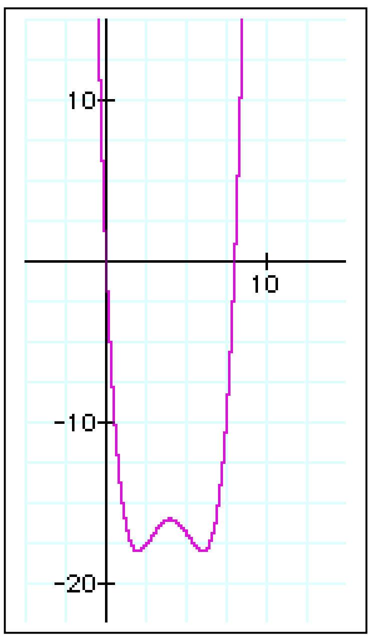
The objective of this problem is to start with a graph showing the rate of change of a function and work backwards to reconstruct the graph of the function itself. In addition to the graph of the rate of change, we will need one value of the function (an initial value) to give us a starting point for sketching the graph of the function.

Initial value of function: $x = 0 \quad y = 0$

Graph of rate of change:



Solution:



The function will be:

- **Increasing:** Over the intervals (2, 4) and (6, ∞).
- **Decreasing:** Over the intervals (−∞, 2) and (4, 6).
- **Concave up:** Over (roughly) the intervals (−∞, 3) and (5, ∞).
- **Concave down:** Over (roughly) the interval (3, 5).

NOTE: Interval notations like (−∞, 3) and (6, ∞) mean:

- (−∞, 3) all numbers less than 3.
- (6, ∞) all numbers greater than 6.

A graph that is consistent with these intervals and which goes through the point $(x, y) = (0, 0)$ appears on the left.

4.5 Example: Using Rates and Concavity to Understand Crime Statistics

During the late 1990's, crime in New York decreased by more than double the national average[2]. Emboldened by these results, New York governor George Pataki made a statement to the press on

[2] Source: Federal Bureau of Investigation. 1997. *Uniform Crime Report*. *1996 Crime Statistics*. Washington DC: Bureau of Justice Statistics.

November 24, 1997. (Note: George Pataki was the first Republican-Conservative governor in the history of New York State.) Portions of this statement are reproduced below[3].

> "Government's top priority must be to safeguard its citizens in our homes, schools and neighborhoods. The FBI's report is yet another example that our policies that keep violent criminals behind bars and off the streets are working. This confirms to the rest of the nation what New Yorkers already know: New York is a safe place to live, work and raise a family ... The Assembly Democrats have repeatedly talked about fighting crime, but has consistently blocked important reforms that will make our streets even safer."

Governor Pataki's comments are based on measurements of the crime **rate** in New York. That is, the number of crimes committed per year. When Governor Pataki said that his policies "are working," he was referring to a drop in the crime **rate**. Governor Pataki's comments are backed up by a lot of statistics and reports[4]. For example, the U.S. Department of Justice keeps a database of violent crimes. Table 1 and Figure 1[5] give the national violent crime rate for the US from 1992 to 2000. The units of this violent crime **rate** are:

$$\text{Units} = \frac{\text{Number of violent crimes experienced by a group of 1000 people}}{\text{Years}}.$$

Year	1992	1993	1994	1995	1996	1997	1998	1999	2000
Violent crime rate	47.9	49.1	51.2	46.1	41.6	38.8	36.0	32.1	27.4

Table 1: Violent crime rate in U.S., 1992-2000.

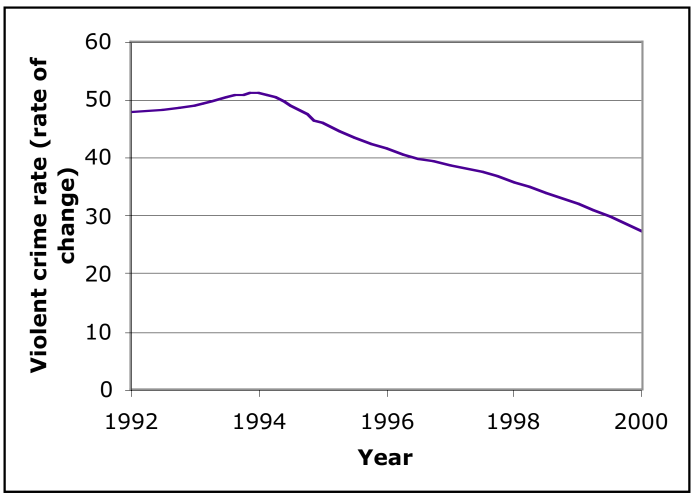


Figure 1: Violent crime rate for United States, 1992-2000. The units of rate are number of violent crimes experienced by a group of 1000 people per year.

One way to relate this information to the mathematics that we have studied is to define a relationship between:

- **Independent variable (*x*):** The amount of time (in years) since January 1, 1992.
- **Dependent variable (*y*):** The total number of violent crimes experienced by a group of 1000 people since January 1, 1992.

---

[3] Source: http://www.state.ny.us/

[4] For example, see: "US violent crime takes sharp drop." available from http://www.cnn.com/ or "Violent crime rate lowest in more than 20 years." also available from http://www.cnn.com/

[5] Sources: 1. U.S. Department of Justice. 2000. *National Crime Victimization Survey*. Washington DC: Bureau of Justice Statistics.
2. Federal Bureau of Investigation. 2001. *Uniform Crime Reports, 1992-2000*. Washington DC: Bureau of Justice Statistics.

The violent crime **rate** described in Table 1 and Figure 1 can be understood as the rate of change of this function.

(a) Use the information given in either Table 1 or Figure 1 to determine the time intervals on which the function is increasing and the time intervals on which the function is decreasing.

(b) Use the information given in either Table 1 or Figure 1 to determine the time intervals on which the function is concave up and the time intervals on which the function is concave down.

(c) When $x = 0$, what value would be reasonable to assume for y?

(d) Sketch a graph showing the total number of violent crimes (y) as a function of years since January 1, 1992 (x).

(e) Over the eight year period from 1992-2000, about how many violent crimes could the average person in the United States expect to be involved with?

(f) Based on the graph that you have sketched, do you think that Governor Pataki's speech of November 24, 1997 is an accurate description of the facts or merely an example of political grandstanding?

Solution

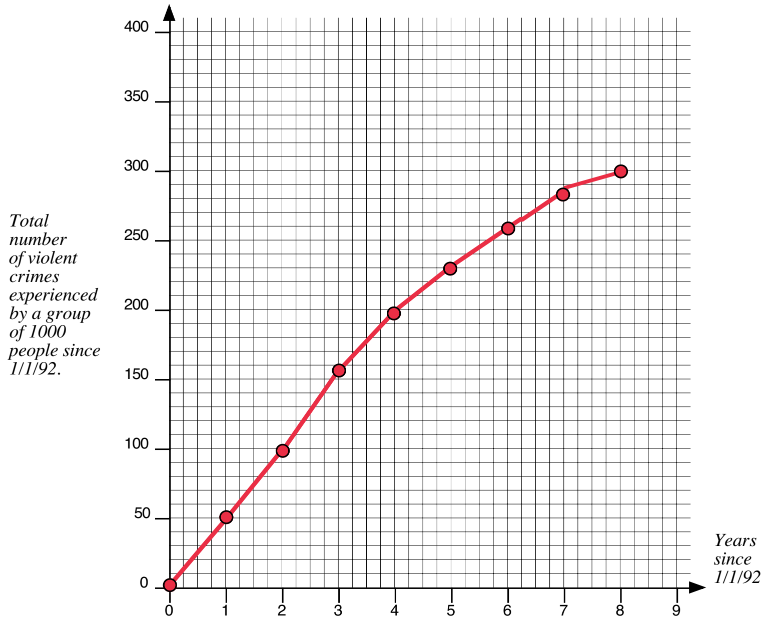
(a) The function will be increasing when the graph of **rate** has positive height and decreasing when the graph of **rate** has negative height. Inspection of the graph in Figure 1 shows that this graph always has positive height, so therefore the function will always be increasing.

(b) The function will be concave up when the graph of **rate** is increasing and concave down when the graph of **rate** is decreasing. Inspecting the graph shown in Figure 1, these intervals will be:

- **Concave up:** (0, 2), i.e. 1992 to 1994.
- **Concave down:** (2, 8), i.e. 1994 to 2000.

(c) It is reasonable to assume that $y = 0$. This is because our function only counts violent crimes committed after midnight on January 1, 1992. Assuming that there wasn't some kind of orgy of violent crime right at the stroke of midnight, when no time has elapsed ($x = 0$) the criminals won't have had any time to commit crimes, so $y = 0$ as well.

(d) A graph that is consistent with all of these observations (and also reasonably consistent with the numerical data in Table 1) is given below.



(e) If the graph that we have sketched is at all accurate, then it predicts that over the eight-year period in question, a group of 1000 people will experience approximately 300 violent crimes. Averaging this by dividing the number of crimes by the number of people, this means that the average person will be involved in 0.3 violent crimes during this eight year period of time.

(f) Governor Pataki cited the FBI report as evidence that his administration's anti-crime policies were working. However, Mr. Pataki's policies only apply to crime committed in New York state, whereas the crime statistics released by the FBI are national averages.

As national averages, the FBI statisticians will certainly have used data on crimes committed in New York state (as well as every other state in the Union) when compiling their national averages. However, national averages do not necessarily reflect the crime rate of New York state. It is entirely possible that the drop in the national violent crime rate that appears in Figure 1 after 1994 could be due to crime reduction efforts in other states. It is, in fact, entirely possible for crime to have risen dramatically in New York state and the national average to go down, so long as a dramatic rise in New York state's crime rate is compensated for by drops in violent crime rates in other states.

Unless he has statistics that measure the violent crime rate in New York state and New York state only, Governor Pataki has no basis in fact for his statements of November 24, 1997. He may well be taking credit for the efforts of people in other states, and it is even possible that his administration's policies may have increased violent crime in New York State. However, as Willie Brown Jr. (the mayor of San Francisco) once said: "If you have a problem taking credit for other peoples' accomplishments, then you've got no business being in politics."

5. Euler's Method

Euler's method is a technique for approximating the values of a function based on just:

- The rate of change of the function.
- One "starting" or "initial" value of the function.
- The width of the time interval (basically the "run" from the rate of change).

The theory behind Euler's method is as follows.

• If you want to get the rate of change of a function, then what you do is select two points from the graph, and find both the rise and run between those points. The rate of change is the rise over the run:

$$Rate = \frac{Rise}{Run}.$$

• If you know the rate and the run, then you can calculate the rise:

$$Rise = (Rate)*(Run).$$

• If you know the value of the function at a point, and you know about how much the independent variable is going to increase (i.e. run) and about how much the dependent variable is going to increase (i.e. the rise) to get to the next point on the graph, then you can calculate the location of the next point on the graph via:

$$\text{next } y = \text{current } y + Rise$$
$$\text{next } x = \text{current } x + Run.$$

• If you keep doing this enough, then you will eventually build up an approximate picture of the values of the function.

5.1 Example: Calculating Crime Statistics

Consider the function that uses **years since 1/1/92** as the independent variable (T), and uses **total number of violent crimes since 1/1/92** as the dependent variable (N). Data on the rate if change of N is given in the table below.

Year	1992	1993	1994	1995	1996	1997	1998	1999	2000
Violent crime rate	47.9	49.1	51.2	46.1	41.6	38.8	36.0	32.1	27.4

Violent crime rate in U.S., 1992-2000.

Approximate the total number of violent crimes that had been committed since 1/1/92 when $T = 10$.

Solution

We will use:

- The fact that when $T = 0$, $N = 0$. (So the initial value of the function is equal to zero.)
- The data on the rate of change given in the table.
- A step-size (or "run") of 2 years.

to calculate the desired value of *N*. Organizing the work into a table (like the one shown below) can help to keep everything well organized.

T	Current value of N	Rate of change	Rise = (rate)×(run)	New value of N
0	0	47.9	95.8	95.8
2	95.8	51.2	102.4	198.2
4	198.2	41.6	83.2	281.4
6	281.4	36.0	72.0	353.4
8	353.4	27.4	54.8	408.2
10	408.2			

So, when $T = 10$, the value of the function will be approximately $N = 408.2$. This means that between 1992 and 2002, a random sample of 1000 people will have experienced about 408.2 violent crimes.