Problem 1 (Fourier Transform of the Gaussian Distribution).

(a) (Changing the Order of Integration for Finite Intervals of Integration).

Let \(-\infty < a < b < \infty\) and \(-\infty < \alpha < \beta < \infty\). Let \(f(x, y)\) be a real-valued continuous function on \([a, b] \times [\alpha, \beta]\). Prove that \(\int_{y=\alpha}^{\beta} f(x, y)\,dy\) is a continuous function of \(x \in [a, b]\) and \(\int_{x=a}^{b} f(x, y)\,dx\) is a continuous function of \(y \in [\alpha, \beta]\) and

\[
\int_{x=a}^{b} \left( \int_{y=\alpha}^{\beta} f(x, y)\,dy \right) \,dx = \int_{y=\alpha}^{\beta} \left( \int_{x=a}^{b} f(x, y)\,dx \right) \,dy,
\]

where all the integrals are in the sense of Riemann integration.

Hint. Choose partitions

\[ a = x_0 < x_1 < x_2 < \cdots < x_{m-1} < x_m = b, \]
\[ \alpha = y_0 < y_1 < y_2 < \cdots < y_{n-1} < y_n = \beta. \]

Let \(m_{\mu, \nu}\) be the infimum of \(f(x, y)\) on \([x_{\mu-1}, x_\mu] \times [y_{\nu-1}, y_\nu]\) and \(M_{\mu, \nu}\) be the supremum of \(f(x, y)\) on \([x_{\mu-1}, x_\mu] \times [y_{\nu-1}, y_\nu]\). Write

\[
\int_{x=a}^{b} \left( \int_{y=\alpha}^{\beta} f(x, y)\,dy \right) \,dx = \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} \int_{x=x_{\mu-1}}^{x_\mu} \left( \int_{y=y_{\nu-1}}^{y_\nu} f(x, y)\,dy \right) \,dx,
\]
\[
\int_{y=\alpha}^{\beta} \left( \int_{x=a}^{b} f(x, y)\,dx \right) \,dy = \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} \int_{y=y_{\nu-1}}^{y_\nu} \left( \int_{x=x_{\mu-1}}^{x_\mu} f(x, y)\,dx \right) \,dy
\]

and compare them with

\[
\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} m_{\mu, \nu} (x_\mu - x_{\mu-1}) (y_\nu - y_{\nu-1}),
\]
\[
\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} M_{\mu, \nu} (x_\mu - x_{\mu-1}) (y_\nu - y_{\nu-1}).
\]

(b) (Changing the Order of Integration for Infinite Intervals of Integration).

For a real-valued continuous function \(g(x)\) on \(\mathbb{R}\) define \(\int_{-\infty}^{\infty} g(x)\,dx\) as the limit \(L\) of

\[
\lim_{a \to -\infty} \int_{a}^{b} g(x)\,dx
\]
if such a limit $L$ exists. In other words, given $\varepsilon > 0$ there exists $A > 0$ such that $\left| \int_a^b g(x) dx - L \right| < \varepsilon$ for all $a < -A$ and $b > A$. More generally, the same definition applies to a function $g(x)$ which is Lebesgue integrable on any finite interval in $\mathbb{R}$.

Let $f(x, y)$ be a real-valued continuous function on $\mathbb{R}^2$. Assume that either

$$\int_{x=-\infty}^{\infty} \left( \int_{y=-\infty}^{\infty} |f(x, y)| dy \right) dx < \infty$$

or

$$\int_{y=-\infty}^{\infty} \left( \int_{x=-\infty}^{\infty} |f(x, y)| dx \right) dy < \infty.$$  

Here, for the first integral $\int_{x=-\infty}^{\infty} \left( \int_{y=-\infty}^{\infty} |f(x, y)| dy \right) dx$, because we do not know whether the function $F(x)$ defined by

$$x \mapsto \int_{y=-\infty}^{\infty} |f(x, y)| dy$$

is a continuous function of $x$, we interpret $F(x)$ as a Lebesgue measurable function of $x$ and the first integral $\int_{x=-\infty}^{\infty} \left( \int_{y=-\infty}^{\infty} |f(x, y)| dy \right) dx$ as the Lebesgue integral of $F(x)$ over $(-\infty, \infty)$ with respect to $x$. For the second integral $\int_{y=-\infty}^{\infty} \left( \int_{x=-\infty}^{\infty} |f(x, y)| dx \right) dy$ we use a similar interpretation.

Verify that

$$\int_{x=-\infty}^{\infty} \left( \int_{y=-\infty}^{\infty} f(x, y) dy \right) dx = \int_{y=-\infty}^{\infty} \left( \int_{x=-\infty}^{\infty} f(x, y) dx \right) dy,$$

where the interpretation in terms of Lebesgue integration is used for the integration with respect to $x$ on the left-hand side and for the integration with respect to $y$ on the right-hand side.

(c) (Differentiation of an Integral with Respect to a Parameter in the Integrand). Let $-\infty < a < b < \infty$ and $g(x, y)$ be a continuous function on $[a, b] \times (-\infty, \infty)$. We say that $\int_{y=a}^{\beta} g(x, y) dy$ is convergent to $\int_{y=-\infty}^{\infty} g(x, y) dy$ uniformly in $x \in [a, b]$ as $\alpha \rightarrow -\infty$ and $\beta \rightarrow \infty$ if for every $\varepsilon > 0$ there exists $A > 0$ such that

$$\left| \int_{y=a}^{\beta} g(x, y) dy - \int_{y=-\infty}^{\infty} g(x, y) dy \right| < \varepsilon$$

for all $a < -A$ and $b > A$ and $x \in [a, b]$.  

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Let \( f(x, y) \) be a continuous function on \([a, b] \times (-\infty, \infty)\) such that \( \frac{\partial f}{\partial x} \) is also continuous on \([a, b] \times (-\infty, \infty)\). Assume that

(i) for each \( x \in [a, b] \) the integral \( \int_{y=\alpha}^{\beta} f(x, y) \, dy \) is convergent to \( \int_{y=-\infty}^{\infty} f(x, y) \, dy \) as \( \alpha \to -\infty \) and \( \beta \to \infty \) and

(ii) \( \int_{y=\alpha}^{\beta} \frac{\partial f(x, y)}{\partial x} \, dy \) is convergent to \( \int_{y=-\infty}^{\infty} \frac{\partial f(x, y)}{\partial x} \, dy \) uniformly in \( x \in [a, b] \) as \( \alpha \to -\infty \) and \( \beta \to \infty \).

Verify that

\[
\frac{d}{dx} \int_{y=-\infty}^{\infty} f(x, y) \, dy = \int_{y=-\infty}^{\infty} \frac{\partial f(x, y)}{\partial x} \, dy
\]

for \( x \in (a, b) \).

(d) (Fourier Transform of the Gaussian Distribution). The Fourier transform \( \hat{f}(\xi) \) of a function \( f(x) \) on \((-\infty, \infty)\) is defined by

\[
\hat{f}(\xi) = \int_{x=-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, dx.
\]

Prove that

\[
\int_{x=0}^{\infty} e^{-x^2} \cos (2xy) \, dx = \frac{1}{2\sqrt{\pi}} e^{-y^2}
\]

by using \( \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \) and by showing that the integral

\[
I(y) := \int_{x=0}^{\infty} e^{-x^2} \cos (2xy) \, dx
\]

satisfies the differential equation \( \frac{dI(y)}{dy} = -2y I(y) \). Hence show that the Fourier transform of \( e^{-\pi x^2} \) is equal to \( e^{-\pi \xi^2} \). More generally, show that for \( \delta > 0 \) the Fourier transform of \( x \mapsto e^{-\pi \delta x^2} \) is \( \xi \mapsto \frac{1}{\sqrt{\delta}} e^{-\pi \xi^2 / \delta} \).

Problem 2 (Approximate Identity). For \( \delta > 0 \) let \( K_\delta(x) \) be a real-valued continuous function on \((-\infty, \infty)\). We say that the family of functions \( K_\delta(x) \) is an approximate identity if the following three conditions are satisfied.

(i) (Nonnegativity) \( K_\delta(x) \geq 0 \) for \( x \in (-\infty, \infty) \) and \( \delta > 0 \).

(ii) (Unit Integral) \( \int_{-\infty}^{\infty} K_\delta(x) \, dx = 1 \) for all \( \delta > 0 \).
(iii) (Integral Outside Any Neighborhood of the Origin Approaching 0) For any \( \eta > 0 \) the integral \( \int_{|x| \geq \eta} K_\delta(x) \, dx \) approaches 0 as \( \delta \to 0 \).

For two functions \( f(x) \) and \( g(x) \) on \( \mathbb{R} \) the convolution \( f \ast g \) of \( f \) and \( g \) is a function on \( \mathbb{R} \) and is defined by

\[
(f \ast g)(x) = \int_{t=-\infty}^{\infty} f(x-t)g(t) \, dt.
\]

For a function \( h(x) \) on \( \mathbb{R} \) and \( p > 0 \) let

\[
\|h\|_{L^p(\mathbb{R})} := \left( \int_{-\infty}^{\infty} |h(x)|^p \right)^{\frac{1}{p}}.
\]

For \( p = \infty \) let

\[
\|h\|_{L^\infty(\mathbb{R})} = \sup_{x \in \mathbb{R}} |h(x)|.
\]

The function \( h(x) \) is said to be \( L^p \) on \( \mathbb{R} \) if \( \|h\|_{L^p(\mathbb{R})} \) is finite.

(a) (Convolution of a Function by Approximate Identity Approaches the Original Function). Let \( f(x) \) be a uniformly continuous function on \( \mathbb{R} \) which is also uniformly bounded. Verify that, for any family \( K_\delta(x) \) of functions which is an approximate identity, the function \( (f \ast K_\delta)(x) \) converges to \( f(x) \) uniformly in \( x \in (-\infty, \infty) \) as \( \delta \to 0 \). In general, for \( 1 \leq p \leq \infty \), if \( f(x) \) is a uniformly continuous function on \( \mathbb{R} \) which is \( L^p \) on \( \mathbb{R} \), then

\[
\|(f \ast K_\delta) - f\|_{L^p(\mathbb{R})} \to 0 \quad \text{as} \quad \delta \to 0.
\]

Hint. For the case \( p = \infty \), write

\[
(f \ast K_\delta)(x) - f(x) = \int_{|t|<\eta} K_\delta(t) (f(x-t) - f(x)) \, dt + \int_{|t|\geq\eta} K_\delta(t) (f(x-t) - f(x)) \, dt
\]

and estimate the first term on the right-hand side by

\[
\left( \int_{|t|<\eta} K_\delta(t) dt \right) \sup_{|t|<\eta} |f(x-t) - f(x)|
\]

and the second term on the right-hand side by

\[
\left( \int_{|t|\geq\eta} K_\delta(t) dt \right) \left( \sup_{|t|\geq\eta} |f(x-t)| + |f(x)| \right).
\]
(b) \textit{(Approximate Identity from the Gaussian Distribution).} Let
\[ K_\delta(x) = \frac{1}{\sqrt{\delta}} e^{-\frac{\pi x^2}{\delta}} \quad \text{for} \quad x \in \mathbb{R} \quad \text{and} \quad \delta > 0. \]
Verify that the family of functions \( K_\delta(x) \) is an approximate identity.

\textbf{Problem 3 (Schwartz Space, Multiplication Formula, Fourier Inversion, and Plancherel Formula).} The Schwartz space \( \mathcal{S}(\mathbb{R}) \) on \( \mathbb{R} \) is defined as consisting of all complex-valued functions \( f(x) \) on \( \mathbb{R} \) such that
\[ \sup_{x \in \mathbb{R}} |x|^k \left| \frac{d^\ell f(x)}{dx^\ell} \right| < \infty \quad \text{for all nonnegative integers} \quad k \quad \text{and} \quad \ell. \]

(a) \textit{(Schwartz Space Closed Under Fourier Transform).} Verify that for \( f \in \mathcal{S}(\mathbb{R}) \) the Fourier transform \( \hat{f} \) of \( f \) also belongs to \( \mathcal{S}(\mathbb{R}) \).

(b) \textit{(Multiplication Formula).} Use Problem 1(b) to show that
\[ \int_{x=-\infty}^{\infty} f(x) \hat{g}(x) \, dx = \int_{y=-\infty}^{\infty} \hat{f}(y) g(y) \, dy \]
for \( f, g \in \mathcal{S}(\mathbb{R}) \), where \( \hat{f} \) is the Fourier transform of \( f \) and \( \hat{g} \) is the Fourier transform of \( g \).

(c) \textit{(Fourier Inversion).} For \( f \in \mathcal{S}(\mathbb{R}) \) verify that
\[ (\dagger) \quad f(0) = \int_{\xi=-\infty}^{\infty} \hat{f}(\xi) \, d\xi \]
(where \( \hat{f} \) is the Fourier transform of \( f \)) by using Part(b) with \( g(x) = e^{-\pi \delta x^2} \) and \( \hat{g}(x) = \frac{1}{\sqrt{\delta}} e^{-\frac{\pi x^2}{\delta}} \) and letting \( \delta \to 0^+ \) and using Problem 2(b). Hence derive the Fourier inversion formula
\[ f(x) = \int_{\xi=-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi \]
by using the function \( y \mapsto f(x+y) \) in \((\dagger)\) whose value at \( y = 0 \) is \( f(x) \).

(d) \textit{(Plancherel Formula).} For \( f \in \mathcal{S}(\mathbb{R}) \), prove the Plancherel formula
\[ \int_{x=-\infty}^{\infty} |f(x)|^2 \, dx = \int_{\xi=-\infty}^{\infty} |\hat{f}(\xi)|^2 \, d\xi \]
(where $\hat{f}$ is the Fourier transform of $f$) by using the following steps. Define $g(x) = f(-x)$ and let $h = f * g$ be the convolution of $f$ and $g$. Verify that $\hat{h}(\xi) = |\hat{f}(\xi)|^2$ (where $\hat{h}$ is the Fourier transform of $h$) and that $h(0) = \int_{x=-\infty}^{\infty} |f(x)|^2 dx$. Then apply (†) to the function $h(x)$.

**Problem 4 (Definite Integrals Evaluated by Using the Beta Function).**

(a) If $\alpha > 0$ and $\beta > 0$ and $x > y$, show that

$$\int_{t=y}^{x} (x-t)^{\alpha-1}(t-y)^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} (x-y)^{\alpha+\beta-1}.$$

**Remark.** The formula is useful in estimating the relation between the composition of the two convolutions, one by $\frac{1}{|x|^{1-\alpha}}$ and the other by $\frac{1}{|x|^{1-\beta}}$, and the single convolution by $\frac{1}{|x|^{1-\alpha-\beta}}$. In the case of $x \in \mathbb{R}^3$, an appropriate constant times the convolution of a function $f$ by $\frac{1}{|x|^\alpha}$ over $\mathbb{R}^3$ is equal to the solution of the Laplace equation whose right-hand side is $f$.

(b) If $\alpha > 0$ and $\beta > 0$ and $x > y$ and either $\lambda < y$ or $\lambda > x$, show that

$$\int_{t=y}^{x} \frac{(x-t)^{\alpha-1}(t-y)^{\beta-1}}{|t-\lambda|^{\alpha+\beta}} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \frac{(x-y)^{\alpha+\beta-1}}{|x-\lambda|^{\beta} |y-\lambda|^\alpha}.$$

**Hint.** For Part(a) apply an appropriate change of variables to the following relation between the beta function and the gamma function

$$\int_{s=0}^{1} (1-s)^{\alpha-1} s^{\beta-1} ds = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

For Part(b) use an appropriate change of variables to reduce it to Part(a).