Notations. $\mathbb{N} =$ all positive integers.
$\mathbb{R} =$ all real numbers.
$\mathbb{C} =$ all complex numbers.

Problem 1 (Cauchy-Riemann equations). Let $m, n \in \mathbb{N}$. Let $U$ an open subset of $\mathbb{C}^n$ and $W$ an open subset of $\mathbb{C}^m$ and $f : U \to W$ be a map. Let $(z_1, \cdots, z_n)$ with $z_j = x_j + \sqrt{-1} y_j$ be the complex coordinates of $\mathbb{C}^n$ which contains $U$ and let $(w_1, \cdots, w_m)$ with $w_j = u_j + \sqrt{-1} v_j$ be the complex coordinates of $\mathbb{C}^m$ which contains $W$ so that

$$u_j = u_j (x_1, \cdots, x_n, y_1, \cdots, y_n), \quad v_j = v_j (x_1, \cdots, x_n, y_1, \cdots, y_n)$$

for $1 \leq j \leq m$ represent the map $f$. Let $P_0 \in U$ and $g : \mathbb{C}^n \to \mathbb{C}^m$ be a map which is $\mathbb{R}$-linear. Assume that $g$ approximates $f$ at $P_0$ to an order higher than the first in the sense that

$$\lim_{P \to P_0} \frac{\| (f(P) - f(P_0)) - g(P - P_0) \|}{\| P - P_0 \|} = 0,$$

where the difference $P - P_0$ of the two points $P_0$ and $P_1$ is considered naturally as a vector in $\mathbb{C}^n$ going from $P_0$ to $P_1$ and $\| \cdot \|$ means the Euclidean norm in the vector space $\mathbb{C}^n$ or $\mathbb{C}^m$. Show that the map $g : \mathbb{C}^n \to \mathbb{C}^m$ is linear over $\mathbb{C}$ if and only if the following Cauchy-Riemann equations

$$\frac{\partial u_j}{\partial x_k} = \frac{\partial v_j}{\partial y_k}, \quad \frac{\partial u_j}{\partial y_k} = - \frac{\partial v_j}{\partial x_k}$$

for $1 \leq j \leq m$ and $1 \leq k \leq n$ are satisfied at the point $P_0$.

Problem 2 (Weierstrass nowhere differentiable function). Let $0 < b < 1$ and $a$ be an odd positive integer such that $ab > 1 + \frac{3\pi}{2}$. Let

$$f(x) = \sum_{n=0}^{\infty} b^n \cos (a^n \pi x).$$

Verify that the continuous function $f(x)$ on $\mathbb{R}$ is nowhere differentiable.
Hint: Fix $x \in \mathbb{R}$. For any $k \in \mathbb{N}$ write $a^k x = n_k + y_k$ uniquely with $-\frac{1}{2} \leq y_k < \frac{1}{2}$ and $n_k \in \mathbb{N}$. Let $h_k = \frac{1 - y_k}{a^k}$. Then

$$\left| \sum_{n=1}^{k-1} \frac{b^n \cos (a^n \pi (x + h_k)) - b^n \cos (a^n \pi x)}{h_k} \right| \leq \frac{\pi (ab)^k}{ab - 1}$$

and the absolute value of the series

$$\sum_{n=k}^{\infty} \frac{b^n \cos (a^n \pi (x + h_k)) - b^n \cos (a^n \pi x)}{h_k}$$

is no less than the absolute value of its first term which is at least $\frac{2}{3} (ab)^k$.

**Problem 3 (Differentiation term-by-term).** Let $-\infty < a < b < \infty$ and $f_k : (a, b) \to \mathbb{R}$ for $k \in \mathbb{N}$. Assume that the second-order derivative $f_k''(x)$ of $f_k$ at $x$ exists for every $x \in (a, b)$ and $k \in \mathbb{N}$. Let $C$ be a positive number and assume that $\sum_{k \in \mathbb{N}} |f_k''(x)| \leq C$ for $x \in (a, b)$. Let $x_0 \in (a, b)$ and assume that the two series $\sum_{k \in \mathbb{N}} f_k(x_0)$ and $\sum_{k \in \mathbb{N}} f_k'(x_0)$ both converge, where $f_k'(x)$ means the first-order derivative of $f_k$ at $x$. Show that the series $\sum_{k \in \mathbb{N}} f_k(x)$ converges at every $x \in (a, b)$ and can be differentiated term-by-term in the sense that the first-order derivative of the function $\sum_{k \in \mathbb{N}} f_k(x)$ at the point $x \in (a, b)$ is equal to $\sum_{k \in \mathbb{N}} f_k'(x)$. Verify that the same conclusion holds when $f_k : (a, b) \to \mathbb{R}$ is replaced by a vector-valued function $f_k : (a, b) \to \mathbb{R}^n$ (where $n \in \mathbb{N}$ is the same for all $k$) and the absolute value $|\cdot|$ is replaced by the norm $\|\cdot\|$. 

**Problem 4 (Chain rule for second-order derivatives).** Let $U$ and $V$ be finite-dimensional $\mathbb{R}$-vector spaces (not necessarily of the same dimension), endowed with norms $\|\cdot\|$, and $D$ be an open subset of $U$ and $G$ be an open subset of $V$. Let $f : D \to G$ and $P_0 \in D$. We say that $f$ is twice differentiable at $P_0$ if there exist an $\mathbb{R}$-linear map $A : U \to V$ and an $\mathbb{R}$-bilinear map $B : U \times U \to V$ such that $f$ is approximated at $P_0$ by $f(P_0) + A(P - P_0) + \frac{1}{2} B(P - P_0, P - P_0)$ to an order higher than the second in the sense that

$$\lim_{P \to P_0} \frac{\|f(P) - (f(P_0) + A(P - P_0) + \frac{1}{2} B(P - P_0, P - P_0))\|}{\|P - P_0\|^2} = 0,$$

where $P - P_0$ is naturally regarded as an element of $U$. We call the $\mathbb{R}$-linear map $A : U \to V$ the first-order derivative of $f$ at $P_0$ and call the $\mathbb{R}$-bilinear
map $B : U \times U \to V$ the second-order derivative of $f$ at $P_0$. Suppose $W$ is a finite-dimensional $\mathbb{R}$-vector space endowed with a norm $\| \cdot \|$ and $H$ is an open subset of $W$ and $g : G \to H$ is a map. Let $Q_0 = f(P_0)$. Assume that $g : D \to H$ is twice differentiable at $Q_0$ with first-order derivative $S : V \to W$ at $Q_0$ and second-order derivative $T : V \times V \to W$ at $Q_0$. Let $h = g \circ f$ so that $h$ maps the open subset $D$ of $U$ to the open subset $H$ of $W$. Denote by $X : U \to W$ the $\mathbb{R}$-linear map $S \circ A$ and denote by $Y : U \times U \to W$ the $\mathbb{R}$-bilinear map $S \circ B + T \circ (A \times A)$, where $A : U \times U \to V \times V$ is defined by

\[ U \times U \ni (P_1, P_2) \mapsto (A(P_1), A(P_2)) \in V \times V. \]

Show that $h$ is twice differentiable at $P_0$ with $X$ as its first-order derivative at $P_0$ and with $Y$ as its second-order derivative at $P_0$.

Problem 5 (Uniqueness of solutions of ordinary differential equations).

(a) Let $k \in \mathbb{N}$ and $-\infty < \alpha_j < \beta_j < \infty$ for $1 \leq j \leq k$ and $-\infty < a < b < \infty$. Let $\phi(x, y_1, \cdots, y_k)$ be an $\mathbb{R}^k$-valued function for $a \leq x \leq b$ and $\alpha_j \leq y_j \leq \beta_j$ for $1 \leq j \leq k$. Let $\vec{c} = (c_1, \cdots, c_k)$ with $\alpha_j \leq c_k \leq \beta_j$ for $1 \leq j \leq k$. Let $\vec{y} = (y_1, \cdots, y_k)$. A solution of the initial-value problem

\[ \vec{y}' = \phi(x, \vec{y}), \quad \vec{y}(a) = \vec{c} \]

is by definition a differentiable vector-valued function $\vec{f}(x) = (f_1(x), \cdots, f_k(x))$ for $a \leq x \leq b$ such that $\vec{f}(a) = \vec{c}$ and $\vec{f}'(x) = \phi(x, \vec{f}(x))$ for $a \leq x \leq b$. Show that such a problem has at most one solution if there exists $A \in \mathbb{R}$ such that

\[ \| \phi(x, \vec{y}) - \phi(x, \vec{z}) \| \leq A \| \vec{y} - \vec{z} \| \]

for $\vec{y} = (y_1, \cdots, y_k)$ and $\vec{z} = (z_1, \cdots, z_k)$ with $\alpha_j \leq y_j \leq \beta_j$ and $\alpha_j \leq z_j \leq \beta_j$ for $1 \leq j \leq k$.

(b) Specialize Part (a) by considering the system

\[ y_j' = y_{j+1} \quad (y = 1, \cdots, k - 1), \]

\[ y_k' = f(x) - \sum_{j=1}^{k} g_j(x)y_j, \]

where $f, g_1, \cdots, g_k$ are continuous $\mathbb{R}$-valued functions on $[a, b]$, and derive a uniqueness theorem for solutions of the equation

\[ y^{(k)} + g_k(x)y^{(k-1)} + \cdots + g_2(x)y' + g_1(x)y = f(x), \]
subject to initial conditions

\[ y(a) = c_1, \quad y'(a) = c_2, \quad \cdots, \quad y^{(k-1)}(a) = c_k. \]

*Hint:* This problem is from #28 and #29 of page 119 of Rudin’s book. See the hints given there.