Peano’s Axioms. The five axioms of Giuseppe Peano are as follows.

1. The set \( \mathbb{N} \) of natural numbers contains an element 1.
2. There is an immediate successor \( x' \in \mathbb{N} \) defined for every element \( x \in \mathbb{N} \).
3. 1 is not an immediate successor of any element of \( \mathbb{N} \).
4. Two distinct elements of \( \mathbb{N} \) have distinct immediate successors.
5. If a subset \( E \) of \( \mathbb{N} \) contains 1 and contains the immediate successor of every one of its elements, then \( E \) must be all of \( \mathbb{N} \).

Problem 1. Use the five axioms of Giuseppe Peano to show that an element \( x \) of \( \mathbb{N} \) is the immediate successor of some element of \( \mathbb{N} \) if and only if it is not equal to 1. (Intuitively, this simply says that \( x - 1 \) is a natural number if and only if \( x \) is a natural number different from 1.)

Problem 2. Addition in \( \mathbb{N} \) is defined by \( x + 1 = x' \) and \( x + y' = (x + y)' \). Use the five axioms of Giuseppe Peano to show that addition is commutative in the sense that \( x + y = y + x \) for any \( x, y \in \mathbb{N} \) according to the definition of addition defined above. (Of course, this is just the commutative law for addition of natural numbers.)

Problem 4. Let \( k \) be a positive integer. Let \( p_1, \ldots, p_k \) be distinct prime numbers and \( n_1, \ldots, n_k \) be positive integers. Let \( r \) be a prime number. Let \( m = p_1^{n_1} \cdots p_k^{n_k} \). Assume that one of \( n_1, \ldots, n_k \) is not divisible by \( r \). Show that the \( r \)-th root of \( m \) is irrational.

Problem 5. Let \( E \) be a countable set of distinct real numbers \( \{a_n\}_{n=1}^{\infty} \) with both an upper bound and a lower bound. For \( k \in \mathbb{N} \) let \( E_k = \{a_n\}_{n=k}^{\infty} \) and \( b_k = \sup E_k \). Let \( F = \{b_k\}_{k=1}^{\infty} \) and \( c = \inf F \). Show that for every \( \varepsilon > 0 \) there exist at most a finite number of elements \( x \) of \( E \) with \( x > c + \varepsilon \) and there exist an infinite number of elements \( y \) of \( E \) with \( y > c - \varepsilon \).

Problem 6. Let \( E, F, G \) be subsets of the set \( \mathbb{R}_{>0} \) of all positive real numbers such that both \( E \) and \( F \) admit an upper bound and \( G \) admits a positive number as a lower bound. Let \( E + F \) be the set of all positive numbers \( x + y \) with \( x \in E \) and \( y \in F \). Let \( E \odot F \) be the set of all real numbers \( x - y \) with
Let $E \cdot F$ be the set of all positive numbers $x \cdot y$ with $x \in E$ and $y \in F$. Let $E \leq F$ be the set of all positive numbers $\frac{x}{z}$ with $x \in E$ and $z \in G$. Verify the following relations.

\[
\begin{align*}
\sup (E + F) &= \sup E + \sup F, \\
\sup (E \ominus F) &= \sup E - \inf F, \\
\sup (E \cdot F) &= (\sup E) (\sup F), \\
\sup \left( \frac{E}{G} \right) &= \frac{\sup E}{\inf G}.
\end{align*}
\]

**Problem 7.** A (Dedekind) cut (of the set $\mathbb{Q}_{>0}$ of all positive rational numbers) is defined as a nonempty proper subset $E$ of $\mathbb{Q}_{>0}$ not containing its least upper bound such that $x < y, y \in E \Rightarrow x \in E$. The product $E \cdot F$ of two cuts $E$ and $F$ is defined as the cut consisting of all elements $x \cdot y$ with $x \in E$ and $y \in F$. Let $1$ be the cut consisting of all rational number $x$ with $0 < x < 1$. Let $E^F$ the cut consisting of all positive rational numbers $z$ with $z < x^y$ for some $x \in E$ and some $y \in F$.

(a) Verify that for every cut $E$ there exists a cut $F$ such that $E \cdot F = 1$.

(b) Verify that for any three cuts $E, F, G$ the two cuts $(E^F)^G$ and $E^{(F \cdot G)}$ are equal.

**Definition of Metrics of Fields.** Let $F$ be a field. A function $\varphi : F \to \mathbb{R}$ is called a metric of the field $F$ if the following properties hold.

1. (positivity) $\varphi(x) > 0$ for $0 \neq x \in F$ and $\varphi(0) = 0$.

2. (triangle inequality) $\varphi(x + y) \leq \varphi(x) + \varphi(y)$.

3. (multiplicativity) $\varphi(xy) = \varphi(x) \varphi(y)$.

A metric $\varphi$ of the field $F$ is called nontrivial if $\varphi(x) \neq 1$ for some $0 \neq x \in F$. A metric $\varphi$ of the field $F$ is called non-Archimedean if the following stronger form of the triangle inequality $\varphi(x + y) \leq \max(\varphi(x), \varphi(y))$ holds for $x, y \in F$. Otherwise, it is called Archimedean. A metric $\varphi$ of a field $F$ is non-Archimedean if and only if $\varphi(n \cdot 1) \leq 1$ for every element of $n \in \mathbb{N}$, where the factor $1$ in $n \cdot 1$ is the unit element of the field $F$ (and $n \cdot 1$ can be alternatively described as the sum of $n$ copies of the unit element $1$ of $F$).
Problem 8. Show that a metric \( \varphi \) of a field \( F \) is non-Archimedean if and only if \( \varphi(n \cdot 1) \leq 1 \) for every element of \( n \in \mathbb{N} \), where the factor 1 in \( n \cdot 1 \) is the unit element of the field \( F \) (and \( n \cdot 1 \) can be alternatively described as the sum of \( n \) copies of the unit element 1 of \( F \)).

Problem 9. Show that every nontrivial Archimedean metric \( \varphi \) of the field \( \mathbb{Q} \) of all rational numbers must of the form \( \varphi(x) = |x|^\gamma \) for some \( 0 < \gamma \leq 1 \).

Hint: Choose \( a \in \mathbb{N} \) such that \( \varphi(a) > 1 \) and choose \( 0 < \gamma \leq 1 \) with \( \varphi(a) = a^\gamma \). For any \( n \in \mathbb{N} \), by expressing \( a = \sum_{j=1}^{k} c_j a^j \) with \( 0 \leq c_j < a \) and applying \( \varphi \) to both sides, show that \( \varphi(n) \leq Cn^\gamma \) for some constant \( C \) independent of \( n \). Replacing \( n \) by \( n^m \) for \( m \) sufficiently large \( m \) shows that \( C \) can be taken to be 1. Applying \( \varphi \) to the special case \( n = a^k - b \) with \( 0 < b \leq a^k - a^{k-1} \) to conclude that \( \varphi(n) \geq C'n^\gamma \) for some constant \( C' \) independent of \( n \). Replacing \( n \) by \( n^m \) for a sufficiently large \( m \) shows that \( C' \) can be taken to be 1.

Problem 10. For every prime number \( p \) and any \( 0 < \theta < 1 \), let \( \varphi_{p,\theta} \) be the metric of the field \( \mathbb{Q} \) of all rational numbers defined by \( \varphi_{p,\theta}(p^{k} \frac{m}{n}) = \theta^k \) for all integers \( m, n, k \) with \( n \neq 0 \) and \( m, n \) both indivisible by \( p \). Show that every nontrivial non-Archimedean metric \( \varphi \) of the field \( \mathbb{Q} \) of all rational numbers must of the form \( \varphi = \varphi_{p,\theta} \) for some prime number \( p \) and some \( 0 < \theta < 1 \).

Hint: Choose a prime number \( p \) with \( \varphi(p) < 1 \). Using the existence of \( a, b \in \mathbb{Z} \) with \( 1 = ap^k + bq^\ell \) for any \( k, \ell \in \mathbb{N} \) and any prime number \( q \neq p \) to show \( \varphi(q) = 1 \). Apply \( \varphi \) to \( p^{k} \frac{m}{n} \).

Problem 11. A norm \( \psi(\cdot) \) which makes \( \mathbb{Q} \) is a normed vector space over \( \mathbb{Q} \) must be of the form \( \psi(x) = C|x| \) for some positive constant \( C \). More precisely, if \( \psi : \mathbb{Q} \rightarrow \mathbb{R}_{\geq 0} \) such that, for \( x, y \in \mathbb{Q} \),

(i) \( \psi(x) \geq 0 \),
(ii) \( \psi(x) = 0 \) if and only if \( x = 0 \),
(iii) \( \psi(\alpha x) = |\alpha| \psi(x) \) for \( \alpha \in \mathbb{Q} \),
(iv) \( \psi(x + y) \leq \psi(x) + \psi(y) \),

then there exists a positive number \( C \) such that \( \psi(x) = C|x| \) for \( x \in \mathbb{Q} \).
Problem 12. (This problem on metrics is taken from one method developed for the comparison of DNA segments.) Let $\mathcal{A}$ be a finite set of objects which we call an alphabet (in practice, the 20-letter amino acid alphabet of proteins). Let $d(a, b)$ be a distance function on $\mathcal{A}$ for $a, b \in \mathcal{A}$ (in practice, the cost of a mutation from $a$ to $b$). Let $g(a)$ be a positive-valued function on $\mathcal{A}$ for $a \in \mathcal{A}$ (in practice, the positive cost of inserting or deleting the letter $a$). For two finite sequences $a = (a_1, a_2, \ldots, a_m)$ and $b = (b_1, b_2, \ldots, b_n)$, not necessarily of the same length, define $D(a, b)$ by induction on $m$ and $n$, as follows.

$$D(a, b) = \begin{cases} 0 & \text{if } m = n = 0, \\ \sum_{k=1}^n g(b_k) & \text{if } m = 0 \text{ and } n > 0, \\ \sum_{j=1}^m g(a_j) & \text{if } m > 0 \text{ and } n = 0, \\ \min(D(a', b) + g(a_m), D(a', b') + d(a_m, b_n), D(a, b') + g(b_n)) & \text{if } m > 0 \text{ and } n > 0, \end{cases}$$

where $a' = (a_1, a_2, \ldots, a_{m-1})$ and $b' = (b_1, b_2, \ldots, b_{n-1})$. Verify that $D(\cdot, \cdot)$ is a metric on the space of all finite sequences of letters of the alphabet $\mathcal{A}$.

Problem 13. (#7 on p.22 of Rudin’s book) Fix $b > 1$, $y > 0$, and prove that there is a unique real $x$ such that $b^x = y$, by completing the following outline. (This $x$ is called the logarithm of $y$ to the base $b$.)

(a) For any positive integer $n$, $b^n - 1 \geq n (b - 1)$.

(b) Hence $b - 1 \geq n \left( b^{\frac{1}{n}} - 1 \right)$.

(c) If $t > 1$ and $n > \frac{b-1}{t-1}$, then $b^{\frac{1}{n}} < t$.

(d) If $w$ is such that $b^w < y$, then $b^{w + \frac{1}{n}} < y$ for sufficiently large $n$; to see this, apply part (c) with $t = y \cdot b^{-w}$.

(e) If $b^w > y$, then $b^{w - \frac{1}{n}} > y$ for sufficiently large $n$.

(f) Let $A$ be the set of all $w$ such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.

(g) Prove that this $x$ is unique.
Problem 14. Let \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \) be complex numbers. Verify the following identity. 
\[
\left| \sum_{k=1}^{n} a_k b_k \right|^2 + \sum_{1 \leq j < k \leq n} |a_j b_k - a_k b_j|^2 = \left( \sum_{j=1}^{n} |a_j|^2 \right) \left( \sum_{k=1}^{n} |b_k|^2 \right).
\]

Use it to prove Schwarz’s inequality
\[
\left| \sum_{k=1}^{n} a_k b_k \right|^2 \leq \left( \sum_{j=1}^{n} |a_j|^2 \right) \left( \sum_{k=1}^{n} |b_k|^2 \right)
\]
and show that Schwarz’s inequality becomes an identity precisely when there exist two complex numbers \( \lambda \) and \( \mu \) not both zero such that \( \lambda a_j + \mu b_j = 0 \) for \( 1 \leq j \leq n \).

For the special case where \( n = 2 \) or \( 3 \) and all \( a_j, b_j \ (1 \leq j \leq n) \) are real, interpret the identity (\*\*) above in terms of the trigonometric identity
\[
\cos^2 \theta + \sin^2 \theta \equiv 1.
\]

Problem 15. Suppose \( z_1, z_2, z_3 \) are complex numbers such that
\[
|z_1| = |z_2| = |z_3|
\]
and
\[
z_1 + z_2 + z_3 = 0.
\]
Prove that
\[
|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|.
\]

Problem 16. The set \( \mathbb{R} \) of real numbers is given the usual Euclidean distance as its metric. Show that a nonempty subset \( G \) of \( \mathbb{R} \) is open if and only if it is a disjoint (at most countable) union of open intervals. In other words, \( G \) is open if and only if there exist \( n \in \mathbb{N} \cup \{ \infty \} \) and \( -\infty \leq a_k < b_k \leq \infty \) for \( 0 \leq k < n \) such that \( G = \bigcup_{0 \leq k < n} (a_k, b_k) \) with \( (a_k, b_k) \) and \( (a_\ell, b_\ell) \) disjoint for \( k \neq \ell \).