Math 272b, homework 2

February 17, 2004

Problem 1. Let $X$ be a smooth $n$-dimensional manifold admitting an immersion $f : X \to \mathbb{R}^{n+1}$. Let $w = 1 + w_1 + \cdots + w_n$ be the total Stiefel-Whitney class of $X$. Show that $(1 + w_1)w = 1$ in $H^2(X; \mathbb{Z}/2)$.

If $X$ is a smooth $n$-dimensional manifold admitting an immersion $f : X \to \mathbb{R}^{n+2}$, show that $(1 + w_1 + w^2 - w_2)w = 1$.

The real projective plane $\mathbb{R}P^2$ admits an immersion in $\mathbb{R}^3$ as Boy’s surface. It is hard to draw – here is a link to a picture of it. Use the existence of Boy’s surface to provide a calculation of $w_2(\mathbb{R}P^2)$.

A smooth map $f : X \to Y$ between smooth manifolds is an immersion if the derivative $f_* : T_X \to TY$ is injective on each fiber $T_xX$, $x \in X$. This is equivalent to saying that each $x \in X$ has a neighborhood $U_x$ such that the restriction of $f$ embeds $U_x$ as a smooth submanifold of $Y$.

Problem 2. Show that $\mathbb{R}P^4$ does not admit an immersion in $\mathbb{R}^6$. (Note that $\mathbb{R}P^4$ is not orientable. It is true, in fact, that every orientable smooth 4-manifold can be immersed in $\mathbb{R}^6$.)

Problem 3. Let $X$ be a smooth 3-manifold. Show that if the tangent bundle has $w_1 = 0$, then $w_2 = 0$ also.

Suggested strategy: Because $H^2(X; \mathbb{Z}/2) = \text{Hom}(H_2(X; \mathbb{Z}/2), \mathbb{Z}/2)$, it is enough to check that the pairing of $w_2$ with every 2-dimensional $\mathbb{Z}/2$-homology class is zero. Now use the fact that every $\mathbb{Z}/2$-homology class in a 3-manifold can be represented as the fundamental class of a (not necessarily orientable) smooth 2-dimensional submanifold, $\Sigma \hookrightarrow X$. The previous exercises are then useful.
Problem 4. Use the fact that \( \pi_4(S^3) = \mathbf{Z}/2 \) to give a classification of complex vector bundles of rank 2 on \( S^5 \).

Remark. We discussed the “clutching construction” in class. Read more about the clutching construction in Hatcher’s “Vector bundles and K-theory” to justify your answer to this question.

If a complex vector bundle \( E \to X \) has \( c_i(E) = 0 \) for all \( i \geq 0 \), does it follow that \( E \) is trivial?

Problem 5. Let \( E \to X \) be a complex vector bundle of rank \( n \) over a simplicial complex \( X \) of dimension at most \( 2n - 1 \). Prove that \( E \) has a section \( s : X \to E \) that is nowhere zero.

Show that every complex vector bundle \( E \to S^4 \) of rank \( r \geq 2 \) has the form \( \mathbf{C}^{r-2} \oplus E' \), where \( E' \) has rank 2 and \( \mathbf{C}^{r-2} \) denotes the trivial bundle of rank \( r - 2 \).

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