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Solution Set 6, Questions 6 and 7.
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Question 6. Suppose $A : V \to V$ is a linear map, and suppose \{v_1, \ldots, v_m\} is a set of non-zero eigenvectors for $A$ with distinct eigenvalues $\lambda_1, \ldots, \lambda_m$. Show that these vectors are linearly independent. (Hint: Use induction on $m$.)

Answer. Let’s use the hint and applying induction on $m$. So start off with the dependence relation
\[ c_1 v_1 + \cdots + c_m v_m = \sum_{i=1}^{m} c_i v_i = 0 \]
for appropriate $c_i \in F$. As we’re trying to prove linear independence, we hope that those $c_i$ are secretly all zero. But for now, let’s assume that at least two of the $c_i$ are non-zero. (Notice that if all but one, say $c_j$, were zero, then $v_j$ would have to be zero, too. But eigenvectors can’t be zero by definition.) Since we’ve got no where else to turn, we apply $A$ remembering to exploit linearity:
\[ A \left( \sum_{i=0}^{m} c_i v_i \right) = \sum_{i=1}^{m} c_i A(v_i) = \sum_{i=1}^{m} c_i \lambda_i v_1 = A(0) = 0. \]
Now we have two equations in the $v_i$. A reasonable thing to do would be to eliminate one of them, say $v_m$. So multiply our first relation through by $\lambda_m$ and subtract. We’re left with
\[ c_1 (\lambda_1 - \lambda_m) v_1 + \cdots + c_{m-1}(\lambda_{m-1} - \lambda_m) v_{m-1} = 0. \]
Now comes the inductive step. By our assumptions $\lambda_i - \lambda_m \neq 0$ for $i \neq m$. Also, at least one $c_i$ is non-zero, leaving $v_1, \ldots, v_{m-1}$ linearly dependent. Rinse and repeat the above procedure to conclude that $v_1, \ldots, v_{m-2}$ are also linearly dependent. In fact, do this until $v_1$ is linear dependent, i.e., until $v_1 = 0$. But wait! That’s a dirty lie. We know that eigenvectors are always non-zero. Moral of the story: the $c_i$ were all zero to begin with and the $v_i$ had to be linearly independent.

Question 7. Let $V = (\mathbb{Z}/2\mathbb{Z})^3$, and consider the linear map $L : V \to V$ given by $L(x, y, z) = (x + y + z, 2y + 3z, 4z)$. Find the eigenvalues of $L$, and find an eigenbasis for $V$. (Hint: Look for likely choices of eigenvalues—I claim that two of them are easy, and the third follows a pattern.)

Answer. Since I can never visualise these things, let’s slap the coefficients into a matrix (as per usual). Then the matrix of $L$ with respect to the standard basis vectors acting on a vector $x \in V$ looks like
\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 2 & 3 \\
0 & 0 & 4
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
\lambda x \\
\lambda y \\
\lambda z
\end{bmatrix}
= \begin{bmatrix}
x + y + z \\
2y + 3z \\
4z
\end{bmatrix}.
\]
We can see from the matrix that $L(e_1) = 1e_1$. Eigenvalue $\lambda_1 = 1$ with vector $v_1 = e_1$. Unfortunately, as is usually the case, the other two aren’t so obvious. Subtracting the two rightmost terms in the three-part equality and reconstructing our matrix we see that
\[
\begin{bmatrix}
1 - \lambda & 1 & 1 \\
0 & 2 - \lambda & 3 \\
0 & 0 & 4 - \lambda
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 0.
\]
Alternatively, $v_2 \in \ker(A - \lambda I)$. We already know what happens when $\lambda = 1$. Smart guessing suggests we try $\lambda = 2, 4$. Then thanks to Gauss-Jordan elimination, we find that when $\lambda = 2$, $v_2 = (1, 1, 0)^t$ and when $\lambda = 4$, $v_4 = (1, 6, 4)^t$. By question 6, we know
that these three are linearly independent. And since we have three linearly independent vectors in a 3-dimensional space, they’ve got to span. Tada.