Last Name: __________________________
First Name: __________________________

Mathematics 21b
Second Midterm
April 26, 1999

Your Section (circle one):

<table>
<thead>
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No calculators are allowed.
Justify your answers carefully (except for question 1).
Except for question 1, no credit can be given for unsubstantiated answers.
1. For each of the following, circle T for true or F for false. No explanation is necessary.

(a) T F If $S, T$ are $n \times n$ matrices such that $ST = TS$, and $\vec{v}$ is an eigenvector of $S$, then $T\vec{v}$ is also an eigenvector of $S$.

(b) T F If $3 + i$ is an eigenvalue of a real $3 \times 3$ matrix $A$, then $A$ is diagonalizable over $\mathbb{C}$.

(c) T F If $A$ is a real $5 \times 4$ matrix, then $AA^T$ is positive definite.

(d) T F If $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are row vectors, then

\[
\det \begin{bmatrix}
-\vec{v}_1 \\
-\vec{v}_2 \\
-\vec{v}_3 \\
-\vec{v}_4 \\
\end{bmatrix} = \det \begin{bmatrix}
-\vec{v}_2 \\
-\vec{v}_3 \\
-\vec{v}_4 \\
-\vec{v}_1 \\
\end{bmatrix}
\]

(e) T F A $2 \times 2$ real matrix $A$ with $\det(A) < 0$ has 2 distinct real eigenvalues.

(f) T F If $\text{tr}(A) > 0$ then in the dynamical system $\frac{d}{dt} \vec{x} = A\vec{x}(t)$, $\vec{0}$ is not asymptotically stable.
2. a) Let

\[ \vec{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \]

Calculate the area of the parallelogram formed by \(\vec{u}\) and \(\vec{v}\) (i.e. two of its sides are equal to \(\vec{u}\) and \(\vec{v}\)).

b) Is the matrix

\[ A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 & 8 \\ 0 & 0 & 9 & 10 & 11 \\ 0 & 0 & 0 & 12 & 13 \\ 0 & 0 & 0 & 0 & 14 \end{bmatrix} \]

diagonalizable over \(R\)? Briefly explain why or why not.
c) Find a matrix with eigenvalues equal to 2, 3, 5, 7.

d) Which of the following matrices $A, B, C, D, E$ has characteristic polynomial
i) $\lambda^4 - 8\lambda^3 - 26\lambda^2 - 88\lambda + 121$?
   Answer: 

ii) $\lambda^4 - 8\lambda^3 - 19\lambda^2 + 98\lambda - 72$?
   Answer: 

For each polynomial there is exactly one correct answer.

\[
A = \begin{bmatrix}
   6 & 1 & 3 & 1 \\
   2 & -2 & 0 & -2 \\
-1 & 4 & 1 & 4 \\
   2 & 3 & 3 & 3
\end{bmatrix}, \quad
B = \begin{bmatrix}
   -5 & 1 & 2 & 6 \\
   8 & -4 & 4 & 1 \\
-2 & 5 & 4 & 0 \\
   3 & -7 & 6 & -3
\end{bmatrix}, \quad
C = \begin{bmatrix}
   0 & -11 & 0 & 0 \\
   1 & -4 & 2 & 1 \\
   0 & 0 & 11 & 0 \\
   0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
D = \begin{bmatrix}
   11 & 0 & 0 & 0 \\
-6 & -1 & 0 & 0 \\
   8 & 2 & 11 & 0 \\
   1 & 3 & 4 & -1
\end{bmatrix}, \quad
E = \begin{bmatrix}
   9 & 4 & 5 & 9 \\
   0 & 2 & 2 & 7 \\
   0 & 0 & 1 & 3 \\
   0 & 0 & 0 & -4
\end{bmatrix}
\]
3. $A$ is a real $n \times n$ matrix such that $A^2 = -I_n$.
   a) Show that $A$ is invertible.

   b) Show that $n$ must be even.

   c) Show that $A$ has no real eigenvalues.
4. Consider a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Suppose the matrix of $T$ with respect to the basis
\[
\begin{bmatrix}
1 & 1 \\
2 & 3
\end{bmatrix}
\begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}
\]. Find the matrix of $T$ with respect to the basis
\[
\begin{bmatrix}
1 & 1 \\
3 & 4
\end{bmatrix}
\].
5. Consider the quadratic form

\[ q(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_3x_1 \]

a) Find a symmetric matrix \( A \) such that

\[ q(\vec{x}) = \vec{x}^T A \vec{x} \]

for all \( \vec{x} \) in \( \mathbb{R}^3 \).

b) Find all the eigenvalues of \( A \) and their algebraic and geometric multiplicities.
c) Is $A$ positive definite? Briefly justify your answer.

d) Find an orthonormal eigenbasis for $A$. 
6. Consider the dynamical system
\[
\frac{dx_1}{dt} = ax_1 - bx_2
\]
\[
\frac{dx_2}{dt} = bx_1 + ax_2
\]

a) Sketch a phase portrait in the case \(a = 1, b = -1\).

b) For each of the four phase portraits below, indicate (by circling the correct answer) whether the constants \(a\) and \(b\) are positive, negative, or zero.

\[
\begin{array}{cccc}
  a > 0 & b > 0 & a > 0 & b > 0 \\
  = 0 & = 0 & = 0 & = 0 \\
  < 0 & < 0 & < 0 & < 0
\end{array}
\]