Last Name: _______________________
First Name: _______________________

Mathematics 21b

Second Exam
April 10, 2001

Your Section (circle one):

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<tr>
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<th>Richard</th>
<th>Richard</th>
<th>Stephanie</th>
<th>Oliver</th>
<th>Daniel</th>
<th>Alexander</th>
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<td>Yang</td>
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The exam will last 90 minutes.
The exam consists of 5 questions, the first worth 12 points, the second worth 8 points and the others each worth 10 points.
No calculators are allowed.
Justify your answers carefully (except in Questions 1 and 2).
For Questions 3–5, no credit can be given for unsubstantiated answers.
Write your final answers in the spaces provided.
(1) True or False (no explanation is necessary).

T  F : If \( V \) is a subspace of \( \mathbb{R}^n \) and \( \vec{x}, \vec{y} \in \mathbb{R}^n \) then \( \| \text{proj}_V \vec{x} - \text{proj}_V \vec{y} \| \leq \|\vec{x} - \vec{y}\| \).

T  F : All shears are orthogonal transformations.

T  F : If a \( 2 \times 2 \) orthogonal matrix \( A \) has exactly one eigenvalue then \( A = \pm I_2 \).

T  F : If a matrix \( A \) has QR-factorisation \( QR \) then \( \ker A = \ker R \).

T  F : If all the columns of an \( n \times n \)-matrix \( A \) are unit vectors then \( |\det A| \leq 1 \).

T  F : If an \( n \times n \)-matrix \( A \) has eigenvalue 2 and an \( n \times n \)-matrix \( B \) has eigenvalue 5, then \( AB \) has eigenvalue 10.
(2) For each of the following values of $d$ state whether or not $\bar{0}$ is an (asymptotically) stable equilibrium of the dynamical system

$$\bar{x}(t + 1) = \begin{pmatrix} 1 & -1 \\ d & 0 \end{pmatrix} \bar{x}(t).$$

[Circle S for “stable” or U for “unstable”.]

**S** **U** : $d = 2$.

**S** **U** : $d = 1$.

**S** **U** : $d = 1/2$.

**S** **U** : $d = 1/8$. 
This page is blank for rough working.
(3) Let $V \subset \mathbb{R}^4$ denote the image of

$$\begin{pmatrix}
1 & 2 \\
1 & 1 \\
1 & 0 \\
1 & 1
\end{pmatrix}.$$ 

(a) Find the matrix of the orthogonal projection onto $V$. 
(b) Find the matrix of orthogonal projection onto $V^\perp$. [HINT: Consider $\text{proj}_V + \text{proj}_{V^\perp}$]
(4) (a) Find the least squares solution for the equations
\[
\begin{align*}
x + y &= 4 \\
y &= 2 \\
x &= -1.
\end{align*}
\]
(b) Calculate the determinant of
\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 & 2 \\
1 & 1 & 3 & 3 & 3 \\
1 & 1 & 4 & 4 & 4 \\
1 & 1 & 1 & 1 & 5
\end{pmatrix},
\]
(c) Calculate the determinant of
\[
\begin{pmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
\end{pmatrix}.
\]
(5) Let $A$ be a $3 \times 5$ matrix and $B$ a $5 \times 3$ matrix with $AB = I_3$.

(a) Explain why $A$ has rank 3 and why $B$ has rank 3.

(b) Show that the non-zero vectors in $\ker A$ are eigenvectors of $BA$. What are the corresponding eigenvalues?
(c) Show that the non-zero vectors in $\text{Im} \ B$ are eigenvectors of $BA$. What are the corresponding eigenvalues?

(d) Find the characteristic polynomial of $BA$. 