Last Name: __________________________
First Name: __________________________

Math 21b
First Midterm
March 7, 2000

Please circle your section:

Christine Taylor  Alex Popa  Bo Cui  Dmitry Tamarkin  John Boller  John Boller
 10 MWF  11 MWF  11 MWF  12 MWF  10 TTh  11:30 TTh

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1. (9 points)

(a) (6 points) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with $n > m$. Show that $\ker(T) \neq \{0\}$.

(b) (3 points) Find an example of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ with $\ker(T) = \{0\}$.
2. (11 points) The points (2, 2), (−1, 1), and (−2, −6) all lie on a circle in \( \mathbb{R}^2 \) with equation \( x^2 + y^2 + cx + dy + e = 0 \).

(a) (3 points) Write the system of linear equations which will determine \( c, d, \) and \( e \).

(b) (6 points) Write the augmented matrix for this system and find its reduced row echelon form.

(c) (2 points) Find all solutions to this system, and identify the center and radius of the circle.
3. (10 points) Let \( v_1 = (1, 1) \) and \( v_2 = (-1, 1) \) be two vectors in \( \mathbb{R}^2 \). Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a linear transformation such that \( T(v_1) = (1, 1) \) and \( T(v_2) = (0, 2) \).

(a) (5 points) Find the matrix for \( T \).

(b) (5 points) Show that \( T \) is a shear.
4. (11 points) Let $A$ be a $3 \times 3$ matrix such that $A^2 = 0$. (That is, the product of $A$ with itself is the zero matrix.)

(a) (4 points) Show that $\text{Im}(A)$ is a subspace of $\text{ker}(A)$.

(b) (4 points) Determine all possible values for $\text{rank}(A)$, and justify your answer(s).

(c) (3 points) Give an example of such a matrix $A$ for each possible rank.
5. (8 points) Show that the three vectors \( v_1 = (2, -3, 4) \), \( v_2 = (2, -5, 2) \), and \( v_3 = (-4, 5, -9) \) all lie in the same plane. What does this say about their linear independence?
6. (11 points) Let \( A = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} \).

(a) (5 points) Find a matrix \( B \) such that \( B^2 = A \).

(b) (4 points) Determine \( \text{rank}(B) \) with justification. \( \text{Hint: It is possible to answer part (b) without having solved part (a).} \)

(c) (2 points) Determine \( B^{17} \).