Mathematics 21b

First Midterm Examination
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Your Section (circle one):

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No calculators are allowed.
1. Consider the system of linear equations

\[
\begin{align*}
3x_3 - 2x_2 &= -3, \\
x_3 + 2x_4 + 4x_1 &= 1, \\
2x_1 + x_2 - x_3 + x_4 &= 2.
\end{align*}
\]

(a) (2 pts.) Write the coefficient matrix and augmented matrix for this system.

\[
A = \begin{bmatrix}
0 & -2 & 3 & 0 \\
4 & 0 & 1 & 2 \\
2 & 1 & -1 & 1
\end{bmatrix}, \quad [A|b] = \begin{bmatrix}
0 & -2 & 3 & 0 & | & -3 \\
4 & 0 & 1 & 2 & | & 1 \\
2 & 1 & -1 & 1 & | & 2
\end{bmatrix}
\]

(b) (5 pts.) Calculate the row-reduced echelon form of the augmented matrix.

\[
\begin{align*}
\begin{bmatrix}
0 & -2 & 3 & 0 & | & -3 \\
4 & 0 & 1 & 2 & | & 1 \\
2 & 1 & -1 & 1 & | & 2
\end{bmatrix} & \rightarrow \begin{bmatrix}
1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & | & 1 \\
0 & -2 & 3 & 0 & | & -3 \\
0 & 1 & -\frac{1}{2} & 0 & | & -\frac{1}{2}
\end{bmatrix} \\
& \rightarrow \begin{bmatrix}
1 & 0 & \frac{1}{4} & \frac{1}{2} & | & \frac{1}{4} \\
0 & 1 & -\frac{3}{2} & 0 & | & \frac{3}{2}
\end{bmatrix}
\end{align*}
\]

(c) (3 pts.) Find the general solution of the linear system. Verify that your answer does in fact satisfy all the equations.

Let \( x_4 = t \)
Let \( x_3 = s \)
Then \( x_2 = \frac{3}{2} + \frac{3}{2}s \)
\( x_1 = \frac{1}{4} - \frac{1}{4}s - \frac{1}{2}t \)

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = \begin{bmatrix}
\frac{1}{4} - \frac{1}{4}s - \frac{1}{2}t \\
\frac{3}{2} + \frac{3}{2}s \\
s \\
t
\end{bmatrix} = \begin{bmatrix}
\frac{3}{2} \\
0 \\
0 \\
0
\end{bmatrix} + s \begin{bmatrix}
-\frac{1}{4} \\
0 \\
0 \\
0
\end{bmatrix} + t \begin{bmatrix}
-\frac{1}{2} \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
E_6.1: \quad 3(s) - 2(\frac{3}{2} + \frac{3}{2}s) = 3s - 3 - 3s = -3
\]
\[
E_6.2: \quad s + 2t + 4(\frac{1}{4} - \frac{1}{4}s - \frac{1}{2}t) = s + 2t + 1 - s - 2t = 1
\]
\[
E_6.3: \quad 2(\frac{1}{4} - \frac{1}{4}s - \frac{1}{2}t) + (\frac{3}{2} + \frac{3}{2}s) - s + t = \frac{1}{2} - \frac{1}{2}s - t + \frac{3}{2} + \frac{3}{2}s - s + t = 2
\]
2. Let $A$ and $B$ be the matrices

$$
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0 \\
\end{bmatrix}.
$$

(a) (4 pts.) Describe $A, B$ geometrically as linear transformations.

\begin{align*}
A \vec{e}_1 &= \vec{e}_1 \\
A \vec{e}_2 &= \vec{0} \\
A \vec{e}_3 &= \vec{e}_3 \\
B \vec{e}_1 &= \vec{e}_1 \\
B \vec{e}_2 &= -\vec{e}_3 \\
B \vec{e}_3 &= \vec{e}_2 \\

A \text{ is projection onto the } xz\text{-plane} \\
B \text{ is rotation by } \frac{\pi}{2} \text{ clockwise in the } yz\text{-plane.}
\end{align*}

(b) (3 pts.) What are the ranks of $A$ and $B$? Is either $A$ or $B$ invertible? Justify your answers.

\begin{align*}
\text{rank}(A) &= 2 \quad \text{since } \text{ref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and hence } \dim(\ker(A)) = 1 \\
\text{and } A \text{ is not invertible.} \\
\text{rank}(B) &= 3 \quad \text{since } \text{ref}(B) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and hence } \dim(\ker(B)) = 0 \\
\text{and } B \text{ is invertible.} \\
\text{(The inverse of a rotation by } \theta \text{ is a rotation by } -\theta. )
\end{align*}

(c) (3 pts.) Do $A$ and $B$ commute? Interpret your result geometrically.

\begin{align*}
AB &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{AB} \neq \text{BA} \\
(AB) \vec{e}_1 &= \vec{e}_1 \\
(AB) \vec{e}_2 &= \vec{0}, \quad -\vec{e}_3 \\
(AB) \vec{e}_3 &= \vec{0} \\
(BA) \vec{e}_1 &= \vec{e}_1 \\
(BA) \vec{e}_2 &= \vec{0} \\
(BA) \vec{e}_3 &= \vec{e}_2 \\
\text{Image } (AB) &= xz\text{-plane} \\
\text{Ker } (AB) &= z\text{-axis} \\
\text{Image } (BA) &= xy\text{-plane} \\
\text{Ker } (BA) &= y\text{-axis}
\end{align*}
3. (a) (1 pt. each) Define: If \( T: \mathbb{R}^n \to \mathbb{R}^m \) is a linear transformation with matrix \( A \), then

- kernel \( \text{Ker}(A) = \{ \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0} \} \)

- image \( \text{Im}(A) = \{ \mathbf{y} \in \mathbb{R}^m \mid \exists \mathbf{x} \in \mathbb{R}^n \text{ such that } A\mathbf{x} = \mathbf{y} \} \)

- rank \( \text{rank}(A) = \# \text{ leading } 1's \text{ in } \text{rref}(A) \)

- span The span of the set of vectors \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_m \} \) is the set of all possible linear combinations of those vectors \( = \{ c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m \mid c_1, \ldots, c_m \text{ are scalars} \} \)

- basis If \( V \) is a subspace of \( \mathbb{R}^n \) and \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_m \} \subseteq V \), then \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_m \} \) is a basis for \( V \) if the \( \mathbf{v}_i \)'s are linearly independent and the \( \mathbf{v}_i \)'s span \( V \).

(b) If \( A \) is a \( 4 \times 5 \) matrix of rank 4,

- (1 pt.) Is \( A \) invertible? Why?
  
  No, only square matrices may be invertible.

- (1 pt.) What can you say about \( \text{ker}(A) \)?
  
  \( \dim(\text{ker}(A)) = 1 \).

- (1 pt.) What can you say about \( \text{im}(A) \)?
  
  \( \dim(\text{im}(A)) = 4 \).

- (2 pts.) What can you say about a linear system whose coefficient matrix is \( A \)?

  The solution set will be a line.
4. Let \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be orthogonal projection to the subspace of \( \mathbb{R}^3 \) with basis consisting of the

single vector \( \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \). [Note: this is not a unit vector!]

Let \( \mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \).

Thus for any \( \mathbf{v} \in \mathbb{R}^3 \), \( \text{proj}_T \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2} \mathbf{w} \)

\( = \frac{\mathbf{v} \cdot \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}}{5} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \)

(a) (4 pts.) Calculate \( T \mathbf{e}_1, T \mathbf{e}_2, T \mathbf{e}_3 \).

\( T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \frac{2}{5} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \frac{4}{5} \begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix} \)

\( T(\mathbf{e}_2) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \frac{2}{5} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \frac{4}{5} \begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix} \)

\( T(\mathbf{e}_3) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ 0 \end{bmatrix} \)

(b) (2 pts.) Find the matrix for \( T \).

\[ A = \begin{bmatrix} \frac{4}{5} & -\frac{2}{5} & 0 \\ -\frac{2}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

(c) (4 pts.) What are the dimensions of \( \ker(T) \) and \( \text{im}(T) \)? Find bases for these subspaces.

\( \ker(T) = \text{span} \left( \begin{bmatrix} \frac{4}{5} \\ -\frac{2}{5} \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{2}{5} \\ \frac{3}{5} \\ 0 \end{bmatrix} \right) \)

\( \text{ref}(A) = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \)

\( \text{dim}(\ker(T)) = 2 \)

basis for \( \ker(T) \) is \( \left\{ \begin{bmatrix} \frac{4}{5} \\ -\frac{2}{5} \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{2}{5} \\ \frac{3}{5} \\ 0 \end{bmatrix} \right\} \)

\( \text{im}(T) = \text{span} \left( \begin{bmatrix} \frac{4}{5} \\ -\frac{2}{5} \end{bmatrix} \right) \)

\( \text{basis for } \text{im}(T) \) is \( \left\{ \begin{bmatrix} \frac{4}{5} \\ -\frac{2}{5} \end{bmatrix} \right\} \)

You must justify your answers to receive full credit.