Math 21b Midterm 1 Solutions - Fall 2001

1. (a) Simple calculation shows that

\[ \text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}. \]

Thus for \( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \) in the kernel of \( T \), we know \( x_1 = -x_3 \) and \( x_2 = 2x_3 \). Thus the kernel of \( T \) is spanned by \( \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \), and in fact this is a basis for the kernel since this spanning set is clearly linearly independent.

(b) The leading 1’s in \( \text{rref}(A) \) were in the first and second columns, so the first and second columns of \( A \) span the image of \( T \). These are clearly linearly independent (since one is not a multiple of the other), so they form a basis for the image of \( T \):

\[
\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -4 \\ 1 \end{bmatrix} \right\}.
\]

(c) \( \text{rank}(A) = \dim(\text{im}(A)) = 2 \).

2. (a) Linear: \( \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \).

(b) Linear: \( \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \).

(c) Not linear because \( T(0) = 1 \) (not 0).

(d) Linear: \( \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \).

3. (a) \( B \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 15 \end{bmatrix} \).

(b) \( (BA)^{-1} = A^{-1}B^{-1} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \).
4. (a) The set $S$ is sketched on the axes below:

(b) $S$ is not a subspace of the vector space $\mathbb{R}^2$ because it is not closed under addition. For example, $\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 6 \\ -1 \end{bmatrix}$ does not satisfy either of the given equations, but $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ satisfies the first one and $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ satisfies the second one.

(c) $S$ has three subsets that are subspaces of $\mathbb{R}^2$. They are

i. The vectors satisfying $2x - y = 0$.

ii. The vectors satisfying $3x + 5y = 0$.

iii. The origin.

5. (a) We want to find $a$ and $b$ such that $a \begin{bmatrix} -1 \\ 4 \end{bmatrix} + b \begin{bmatrix} -2 \\ 9 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$. Solving the system yields

$$a = -59 \text{ and } b = 26.$$  Thus $T(\begin{bmatrix} 7 \\ -2 \end{bmatrix}) = -59(t^2 - 3) + 26(t + 1) = -59t^2 + 26t + 203$.

(b) $3t^2 + 4t - 5 = 3(t^2 - 3) + 4(t + 1)$. Thus

$$\begin{bmatrix} a \\ b \end{bmatrix} = 3 \begin{bmatrix} -1 \\ 4 \end{bmatrix} + 4 \begin{bmatrix} -2 \\ 9 \end{bmatrix} = \begin{bmatrix} -11 \\ 48 \end{bmatrix}.$$  

(c) The quadratic $t^2$ is not in the image of $T$ because there is no linear combination of $t^2 - 3$ and $t + 1$ that equals $t^2$.

6. (a) The diagram given shows that $T(\vec{w})$ and $T(\vec{v})$ are linearly independent. And since $\text{dim}(\text{im}(T)) \leq 2$, the vectors $T(\vec{v})$ and $T(\vec{w})$ must form a basis for the image of $T$.

(b) Since $T$ is linear, $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = T(\vec{w}) - T(\vec{v})$.

(c) Again since $T$ is linear, $T(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = T(\vec{v}) + T(\vec{w})$. This can be drawn on the axis provided by adding the two vectors to get a vector whose tip is at approximately $(8, 8)$ if we consider the grid marks to be the units.

7. The linear transformation $T$ is from $\mathbb{R}^2$ to $\mathbb{R}^2$, so the matrix $A$, representing $T$, is a $2 \times 2$ matrix. Recall that $\det(A) = (\det(A^{-1}))^{-1}$. Since $A = A^{-1}$, this means $\det(A) = \pm 1$. If $\det(A) = 1$, then
\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}. \]

Thus we get that \( a = d, b = c = 0, \) and \( ad = 1. \) Therefore the only possibilities for \( A \) are \( \pm I_2. \) However, if \( \det(A) = -1, \) then \( A \) must have the form:

\[ A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \]

where \( -a^2 - bc = -1. \) But notice that this equation has infinitely many solutions, since we can rewrite it as:

\[-bc = a^2 - 1 = (a - 1)(a + 1),\]

so any set of values \((a, b, c) = (a, 1 - a, 1 + a)\) will be valid. Thus any matrix of the form

\[ A = \begin{bmatrix} a & 1 - a \\ 1 + a & -a \end{bmatrix} \]

is its own inverse. Thus there are infinitely many such matrices, and so infinitely many self-inverting linear transformations.

(Note: this does not characterize all matrices whose square is the identity, but it sufficiently shows that there are infinitely many.)