1. [6 points] A matrix $M$ is of the form

\[
\begin{bmatrix}
0 & * & 5 & * \\
* & * & * & * 
\end{bmatrix},
\]

where as usual the *'s denote unknown and possibly different real numbers. Given that $M$ is in row-reduced echelon form, find all possible $M$, and explain why there are no other possibilities. For each of the $M$ that you have found, determine its rank, image, and kernel.

Given $M$ is 2x4, with $a_{11} = 0$ and $a_{13} = 5$, and $M$ in row-reduced form.

Since the first row is not all zeroes, it must have a leading 1 before the non-zero entry.

Hence $a_{12} = 1$.

Every other entry in a column with a leading 1 must be zero.

Hence $a_{22} = 0$.

If $a_{21}$ were non-zero, then we could divide the second row by that entry and have a leading 1, but it would be below and to the left of $a_{12}$, which is not row-reduced form.

Hence $a_{21} = 0$.

If $a_{23}$ were non-zero, then it must be a leading 1 since it would be the first non-zero entry of the second row. But then $a_{13}$ would be zero since it is in the same column as a leading 1.

Hence $a_{23} = 0$.

Thus far, we have $M = \begin{bmatrix} 0 & 1 & 5 & * \\ 0 & 0 & 0 & * \end{bmatrix}$

- If $a_{24}$ is a leading 1, then $a_{14}$ is zero, and we have our first possibility:

  $M_1 = \begin{bmatrix} 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

  rank $(M_1) = 2$

  image $(M_1) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2$

  ker $(M_1) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

- If $a_{24}$ is zero, then $a_{14}$ may be anything, say $n$. This is our second possibility:

  $M_2 = \begin{bmatrix} 0 & 1 & 5 & n \\ 0 & 0 & 0 & 0 \end{bmatrix}$

  rank $(M_2) = 1$  

  image $(M_2) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

  ker $(M_2) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

(If $a_{24}$ is anything non-zero, then it must be a leading 1. Thus we have all possibilities.)
2. Consider the matrix

\[
A = \begin{bmatrix}
2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{bmatrix}.
\]

a) [4 points] Construct an orthonormal eigenbasis for \(A\).

\[
\lambda I - A = \begin{bmatrix}
\lambda - 2 & -1 & -1 \\
-1 & \lambda - 2 & -1 \\
-1 & -1 & \lambda - 2
\end{bmatrix}
\]

\[
\det (\lambda I - A) = (\lambda - 2)^3 - 1 - 3(\lambda - 2)
= \lambda^3 - 6\lambda^2 + 9\lambda - 4
= (\lambda - 1)(\lambda^2 - 5\lambda + 4)
= (\lambda - 1)^2(\lambda - 4)
\]

\(A\) has eigenvalues \(1\) (alg. mult. 2) and \(4\) (alg. mult. 1).

\[
\text{rref} (1 \cdot I - A) = \begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad E_1 = \ker (1 \cdot I - A) = \text{span} \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}
\]

\[
\text{rref} (4 \cdot I - A) = \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix}, \quad E_4 = \ker (4 \cdot I - A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}
\]

Then \(\{\vec{w}_1, \vec{w}_3, \vec{w}_4\}\) is an orthonormal eigenbasis for \(A\), but it is not orthonormal.

Since \(A\) is symmetric, it will have an orthonormal eigenbasis. Let \(\vec{v}_1 = \frac{\vec{w}_1}{\|\vec{w}_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\).

Let \(\vec{v}_2 = \vec{w}_2 - (\vec{w}_2 \cdot \vec{v}_1) \vec{v}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\).

Then \(\{\vec{v}_1, \vec{v}_2, \vec{w}_3\}\) is an ONB for \(E_1\).

b) [2 points] Is \(A\) diagonalizable? Why or why not?

Yes, since we have constructed an (orthonormal) eigenbasis. Let \(S = [\vec{v}_1 \quad \vec{v}_2 \quad \vec{w}_3]\).

Then \(S^{-1}AS = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 4
\end{bmatrix}\).

(All symmetric matrices are diagonalizable.)

Let \(\vec{v}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\).

Then \(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}\) is an orthonormal eigenbasis for \(A\).
c) [4 points] Using parts (a) and (b), compute $A^{2000}$.

$$S = \begin{bmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix}$$

$$B = S^{-1}AS = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 4
\end{bmatrix}$$

$$B^{2000} = (S^{-1}AS)^{2000} = S^{-1}A^{2000}S$$

$$A^{2000} = SB^{2000}S^{-1}$$

(since $S$ is orthogonal)

$$= SB^{2000}S^T$$

$$= \begin{bmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 4^{2000}
\end{bmatrix} \begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{2}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 4^{2000}
\end{bmatrix}$$

$$= \begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{2}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 4^{2000}
\end{bmatrix}$$

$$= \begin{bmatrix}
\frac{1}{2} + \frac{1}{6} + \frac{1}{3} \cdot 2^{4000} & -\frac{1}{2} + \frac{1}{6} + \frac{1}{3} \cdot 2^{4000} & -\frac{2}{6} + \frac{1}{3} \cdot 2^{4000} \\
-\frac{1}{2} + \frac{1}{6} + \frac{1}{3} \cdot 2^{4000} & \frac{1}{2} + \frac{1}{6} + \frac{1}{3} \cdot 2^{4000} & -\frac{2}{6} + \frac{1}{3} \cdot 2^{4000} \\
-\frac{3}{6} + \frac{1}{3} \cdot 2^{4000} & -\frac{2}{6} + \frac{1}{3} \cdot 2^{4000} & \frac{4}{6} + \frac{1}{3} \cdot 2^{4000}
\end{bmatrix}$$

4
3. Let $B$ be the matrix $\det B = \det(C) \cdot \det(F)$

$$B = \begin{bmatrix}
1.2 & -0.4 & 0 & 0 \\
1.3 & 0.4 & 0 & 0 \\
0 & 0 & 1.2 & 0.4 \\
0 & 0 & -2.6 & 0.8
\end{bmatrix} = \begin{bmatrix}
C \\
F
\end{bmatrix}$$

a) [3 points] Find all eigenvalues of $B$.

$$\lambda \mathbf{I} - B = \begin{bmatrix}
\lambda - 1.2 & 0.4 \\
-1.3 & \lambda - 0.4 \\
2.6 & \lambda - 0.8
\end{bmatrix}$$

$$\det (\lambda \mathbf{I} - B) = \begin{vmatrix}
(\lambda - 1.2)(\lambda - 0.4) + (0.4)(1.3) \\
\lambda^2 - 1.6 \lambda + 0.48 + 0.52 \\
\lambda^2 - 2 \lambda + 0.96 + 1.04
\end{vmatrix}$$

$$= (\lambda^2 - 1.6 \lambda + 1)(\lambda^2 - 2 \lambda + 2)$$

Eigenvalues:

$$\lambda_1 = \frac{1.6 \pm \sqrt{2.56 - 4}}{2} = 0.8 \pm 0.6 \text{i}$$

$$\lambda_2 = \frac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm \text{i}$$

b) [3 points] Does the dynamical system

$$\vec{x}(t+1) = B \vec{x}(t)$$

have a point of stable equilibrium? Why or why not?

This is a discrete dynamical system.

Stable equilibria occur when the moduli of all eigenvalues are less than 1.

This is not the case here because

$$|\lambda_1| = 1 \quad \text{and} \quad |\lambda_2| = \sqrt{2}$$
c) [4 points] Describe qualitatively the behavior of this dynamical system if \( \mathbf{x}(0) \) is the unit vector
\[
\begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix}
\]

\( \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \) may be written as a linear combination of the eigenvectors for \( \lambda_1 \) and \( \lambda_2 \) since its third and fourth components are zero.

Since \( |\lambda_1| = 1 \) and \( |\lambda_2| = 1 \),
the trajectory of \( \mathbf{x}(0) \) will be an ellipse.
4. Consider the linear transformation $T : C^\infty \to C^\infty$ given by

$$T(f) = f'' - 2f'.$$

(a) [6 points] Find all real eigenvalues of $T$ and their corresponding eigenspaces.

Let $\lambda \in \mathbb{R}$ and suppose $\lambda$ is an eigenvalue for $T$.

Then $T(f) = \lambda f$ for some $f \in C^\infty$.

Support $f'' - 2f' = \lambda f$

$f'' - 2f' - \lambda f = 0$

This is equivalent to finding the kernel of the linear differential operator $T - \lambda : C^\infty \to C^\infty$

$(T - \lambda)f = f'' - 2f' - \lambda f$.

The char. poly. of $T - \lambda$ is:

$P_{T-\lambda}(x) = x^2 - 2x - \lambda$

This has roots

$$\chi = \frac{2 \pm \sqrt{4 + 4\lambda}}{2} = 1 \pm \sqrt{1 + \lambda}$$

These roots are distinct unless $\lambda = -1$.

When the roots of the char. poly. are distinct, we know

$$\ker (T - \lambda) = \text{span} \left\{ e^{(1+\sqrt{1+\lambda})t}, e^{(1-\sqrt{1+\lambda})t} \right\}$$

If $\lambda > -1$, then these two basis functions are real, and we have

$E_{\lambda} = \ker (T - \lambda) = \text{span} \left\{ e^{(1+\sqrt{1+\lambda})t}, e^{(1-\sqrt{1+\lambda})t} \right\}$.

If $\lambda < -1$, then we use Euler's formula:

$$e^{(1+i\sqrt{1-\lambda})t} = e^t (\cos \sqrt{1-\lambda} t + i \sin \sqrt{1-\lambda} t)$$

$$e^{(1-i\sqrt{1-\lambda})t} = e^t (\cos \sqrt{1-\lambda} t - i \sin \sqrt{1-\lambda} t)$$

and we have

$E_{\lambda} = \ker (T - \lambda) = \text{span} \left\{ e^t \cos \sqrt{1-\lambda} t, e^t \sin \sqrt{1-\lambda} t \right\}$.

Special case: $\lambda = -1$.

$$(T - 1)f = f'' - 2f' + f$$

$$(T - 1)f = 0$$

when $f(t) = A t \cdot e^t + B \cdot e^t$, and $E_{-1} = \text{span} \left\{ t \cdot e^t, e^t \right\}$.

In conclusion, every $\lambda \in \mathbb{R}$ is an eigenvalue of $T$ with a two-dimensional eigenspace $E_{\lambda}$. 

b) [2 points] Let $T_2$ be the same linear transformation restricted to the subspace $P_2$ of $C^\infty$, consisting of polynomials of degree at most 2. (That is, $T_2 : P_2 \to P_2$ is the transformation taking any polynomial $f$ of degree at most 2 to $f'' - 2f'$.)

Choose a basis for $P_2$, and write the matrix $A_2$ of $T_2$ with respect to this basis.

$$ P_2 \text{ has a natural basis } \{1, x, x^2\}.$$ 

$T_2(1) = 0$ 

$T_2(x) = -2$ 

$T_2(x^2) = 2 - 4x$ 

With respect to this basis, then, identifying $1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $x^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$ A_2 = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix} $$


c) [4 points] Find the image and kernel of this matrix $A_2$. Check (part of) your work by explaining the relationship between parts (a) and (c).

$$ \text{rref} \left( A_2 \right) = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} $$

$$ \text{ker} \left( A_2 \right) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} = \text{span} \left\{ 1 \right\} $$

$$ \text{image} \left( A_2 \right) = \text{span} \left\{ \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix} \right\} = \text{span} \left\{ -2, 2 - 4x \right\} $$

Any functions in $\ker \left( A_2 \right) = \ker \left( T_2 \right)$ should also be in $\ker \left( T \right)$, i.e. have eigenvalue 0.

We know from part (a) that $\ker \left( T \right) = \text{span} \left\{ e^{2t}, e^{0t} \right\} = \text{span} \left\{ e^{2t}, 1 \right\}$.

This checks for constant functions, but $e^{2t} \not\in P_2$. 

7
5. The following continuous dynamical system models the populations \( x(t), y(t) \) of two species:

\[
\begin{align*}
\frac{\partial x}{\partial t} &= (y - 1)x \\
\frac{\partial y}{\partial t} &= (x + y - 3)y
\end{align*}
\]

We consider only the first quadrant \( x \geq 0, y \geq 0 \) since we are modelling populations.

a) [2 points] Sketch the nullclines of this system, and find any equilibrium points.

b) [4 points] Use linear approximation to find the Jacobian \( J \) of the system at any equilibrium points from part (a).

\[
J = \begin{bmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{bmatrix} = \begin{bmatrix}
(y-1)x \\
(x+y-3)y
\end{bmatrix}
\]

\[
\frac{\partial f}{\partial x} = y - 1 \quad \frac{\partial f}{\partial y} = x
\]

\[
\frac{\partial g}{\partial x} = y \quad \frac{\partial g}{\partial y} = x + 2y - 3
\]

\[
J = \begin{bmatrix}
\frac{\partial f}{\partial x}(a,b) & \frac{\partial f}{\partial y}(a,b) \\
\frac{\partial g}{\partial x}(a,b) & \frac{\partial g}{\partial y}(a,b)
\end{bmatrix} \begin{bmatrix}
x - a \\
y - b
\end{bmatrix}
\]
\( J(\mathbf{P}) = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \)

\( J(\mathbf{Q}) = \begin{bmatrix} 2 & 0 \\ 3 & 3 \end{bmatrix} \)

\( J(\mathbf{R}) = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \)

\[
\text{Generally,} \quad \text{det} = \left( \frac{\lambda}{2} \right)^2 \\
\text{tr} \quad \lambda_1 + \lambda_2
\]

c) [4 points] Analyze the stability of each equilibrium point by computing the eigenvalues of \( J \).

\( J(\mathbf{P}) \) has eigenvalues \(-1, -3\) since it is diagonal.

Both are negative and real, so \( \mathbf{P} \) is a point of stable equilibrium.

\[
\text{det} J(\mathbf{P}) = 3 < \left( -\frac{4}{2} \right)^2, \quad \text{so} \quad \Box \Box \Box
\]

\( J(\mathbf{Q}) \) has eigenvalues \(2, 3\) since it is lower triangular.

Both are positive and real, so \( \mathbf{Q} \) is not a point of stable equilibrium.

\[
\text{det} (\mathbf{Q}) = 6 < \left( \frac{5}{2} \right)^2, \quad \text{so} \quad \Box \Box \Box
\]

\( J(\mathbf{R}) \) has char. poly. \( f_{J(\mathbf{R})}(\lambda) = \lambda^2 - \lambda - 2 \)

\[ = (\lambda - 2)(\lambda + 1) \]

and hence has eigenvalues \(-1, 2\).

With one positive and one negative, \( \mathbf{R} \) is not a point of stable eq.
d) [2 points] Sketch (approximately) the phase plane of this system, including behavior near equilibrium points and approximate direction of the flow lines in the regions separated by the nullclines.

\[
\begin{align*}
\frac{dx}{dt} &= (y-1)x \\
\frac{dy}{dt} &= (x+y-3)y
\end{align*}
\]

<table>
<thead>
<tr>
<th>Region</th>
<th>Pt.</th>
<th>(\frac{dx}{dt})</th>
<th>(\frac{dy}{dt})</th>
<th>Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>(1, \frac{1}{2})</td>
<td>-</td>
<td>-</td>
<td>→</td>
</tr>
<tr>
<td>II</td>
<td>(\frac{1}{2}, 2)</td>
<td>+</td>
<td>-</td>
<td>↓</td>
</tr>
<tr>
<td>III</td>
<td>(2, 2)</td>
<td>+</td>
<td>+</td>
<td>↗</td>
</tr>
<tr>
<td>IV</td>
<td>(3, \frac{1}{3})</td>
<td>-</td>
<td>+</td>
<td>↘</td>
</tr>
</tbody>
</table>
6. Consider the temperature \( T(x,t) \) of a metal bar extending from \( x = 0 \) to \( x = \pi \). The temperature satisfies the heat equation

\[
\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}
\]

and the ends are held at a constant temperature of zero, i.e. \( T(0,t) = 0 \) and \( T(\pi,t) = 0 \) for all \( t \geq 0 \).

a) [3 points] Show that the functions \( e^{-\mu n^2 t} \sin(nx) \) satisfy the differential equation and the initial conditions for all positive integers \( n \).

Let \( T_n(x,t) = e^{-\mu n^2 t} \sin(nx) \).

Then \( \frac{\partial T_n}{\partial t} = -\mu n^2 e^{-\mu n^2 t} \sin(nx) \).

\[
\frac{\partial T_n}{\partial x} = n \cdot e^{-\mu n^2 t} \cos(nx)
\]

\[
\frac{\partial^2 T_n}{\partial x^2} = -n^2 e^{-\mu n^2 t} \sin(nx)
\]

Hence

\[
\frac{\partial T_n}{\partial t} - \mu \frac{\partial^2 T_n}{\partial x^2} = -\mu n^2 e^{-\mu n^2 t} \sin(nx) - \mu \left( -n^2 e^{-\mu n^2 t} \sin(nx) \right)
\]

\[= 0.\]

Also, \( T_n(0,t) = e^{-\mu n^2 t} \sin(0) = 0 \)

\( T_n(\pi,t) = e^{-\mu n^2 t} \sin(n\pi) = 0 \) since \( n \in \mathbb{N} \).
b) [7 points] Suppose now that the initial temperature of the bar is given by

\[ T(x, 0) = \theta(x) = \sin^3 x. \]

Determine \( T(x, t) \) for all \( x \in [0, \pi] \) and all times \( t \geq 0 \). [Hint: Use Euler's formula \( e^{iy} = \cos y + i\sin y \) if you are not sure about the relevant trigonometric identities.] Examine the behavior of \( T \) as \( t \to \infty \).

We expect a solution of the type:

\[ T(x, t) = \sum_{n=1}^{\infty} b_n \cdot e^{-4n^2 t} \cdot \sin nx. \]

The initial condition would need:

\[ T(x, 0) = \sum_{n=1}^{\infty} b_n \cdot \sin nx. \]

So we hope that the first expression is the Fourier series of \( \sin^3 x \).

We hope first for a finite linear combination of these functions that will satisfy the initial condition. (In fact, since \( \sin^3 x \) is a nicely behaved periodic function composed of the basic trigonometric functions, its F. series is finite.)

Euler's Formula:

\[ e^{3ix} = e^{i(3x)} = \cos 3x + i \sin 3x. \]

\[ e^{3ix} = (e^{ix})^3 = (\cos x + i \sin x)^3 \]

\[ = \cos^3 x + 3i \cos^2 x \sin x \]

\[ - 3 \cos x \sin^2 x - 3i \sin^3 x \]

\[ = (\cos^3 x - 3 \cos x \cdot \sin^2 x) \]

\[ + i(3 \cos^2 x \sin x - \sin^3 x) \]

Matching imaginary parts yields:

\[ \sin 3x = 3 \cos^2 x \cdot \sin x - \sin^3 x \]

\[ = 3(1 - \sin^2 x) \sin x - \sin^3 x \]

\[ = 3 \sin x - 4 \sin^3 x \]

and hence \( \sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x \) (This is the F. series for \( \sin^3 x \) since F. curves are unique!)

We let \( b_1 = \frac{3}{4} \) and \( b_3 = -\frac{1}{4} \).