ASYMPTOTIC STABILITY COMPARISON OF DISCRETE AND CONTINUOUS SITUATION.

The trace and the determinant are independent of the basis, they can be computed fast, and are real if \( A \) is real. It is therefore convenient to determine the region in the \( tr - det \)-plane, where continuous or discrete dynamical systems are asymptotically stable. While the continuous dynamical system is related to a discrete system, it is important not to mix these two situations up.

**Continuous dynamical system.**

Stability of \( \ddot{x} = Ax \) \( (x(t + 1) = e^{At}x(t)) \).

- Stability in \( det(A) > 0, tr(A) > 0 \) Stability if \( Re(\lambda_1) < 0, Re(\lambda_2) < 0 \).
- Stability in \( det(A) < 0, tr(A) > 0 \) Stability if \( |\lambda_1| < 1, |\lambda_2| < 1 \).

**Discrete dynamical system.**

Stability of \( x(t + 1) = Az \).

- Stability in \( \lambda_1 < 0, \lambda_2 < 0 \).
- \( i \times A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \).
- \( i \times A = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \).

**PHASE-PORTRIATS.** (In two dimensions we can plot the vector field, draw some trajectories)

- \( \lambda_1 > 0, \lambda_2 > 0 \)
- \( i \times A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \).
- \( i \times A = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} \).

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**The differential equations of the spinner.**

\( x \) is the angle and \( y \) the height of the body. We put the coordinate system so that \( y = 0 \) is the point, where the body stays at rest if \( x = 0 \). We assume that if the spring is wound up with an angle \( x \), this produces an upwards force \( x \) and a momentum force \( -3x \). We furthermore assume that if the body is at position \( y \), then this produces a momentum \( y \) onto the body and an upwards force \( y \).

The differential equations:

\[
\ddot{x} = -3x + y \\
\ddot{y} = -y + x
\]

Finding good coordinates \( w = S^{-1}x \) is obtained by getting the eigenvalues and eigenvectors of \( A \):

\[
\lambda_1 = -2 - \sqrt{2}, \lambda_2 = -2 + \sqrt{2}
\]

\[
v_1 = \begin{bmatrix} -1 - \sqrt{2} \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 + \sqrt{2} \\ 1 \end{bmatrix}
\]

so that \( S = \begin{bmatrix} -1 - \sqrt{2} & -1 + \sqrt{2} \\ 1 & 1 \end{bmatrix} \).

Solve the system \( a = \lambda_1 a, b = \lambda_2 b \) in the good coordinates:

\[
a(t) = A \cos(\omega_1 t) + B \sin(\omega_1 t), \omega_1 = \sqrt{-\lambda_1} \\
b(t) = C \cos(\omega_2 t) + D \sin(\omega_2 t), \omega_2 = \sqrt{-\lambda_2}
\]

The solution in the original coordinates:

\[
S \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}.
\]

At \( t = 0 \) we know \( x(0), y(0), \dot{x}(0), \dot{y}(0) \). This fixes the constants in \( x(t) = A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t) + A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t) \). The curve \( (x(t), y(t)) \) traces a Lyssajoux curve:

ASYMPTOTIC STABILITY \( x = Ax \) is asymptotically stable if and only if all eigenvalues have negative real part \( Re(\lambda) < 0 \).

ASYMPTOTIC STABILITY IN 2D A linear system \( x = Ax \) in the 2D plane is asymptotically stable if and only if \( det(A) > 0 \) and \( tr(A) < 0 \).

Proof: If both eigenvalues \( \lambda_1, \lambda_2 \) are real, then both being negative is equivalent to \( \lambda_1 \lambda_2 = det(A) > 0 \) and \( tr(A) = \lambda_1 + \lambda_2 < 0 \). If \( \lambda_1 = a + ib, \lambda_2 = a - ib \), then a negative \( i \) is equivalent to \( \lambda_1 + \lambda_2 = 2a < 0 \) and \( \lambda_1 \lambda_2 = a^2 + b^2 > 0 \).