DETERMINATOR II, (Judgement day) Math 21b, O.Knill

HOMEWORK: Section 6.2: 6,8,16,18,44*, Section 6.3: 14

DETERMINANT AND VOLUME. If \( A \) is a \( n \times n \) matrix, then \( |\det(A)| \) is the volume of the \( n \)-dimensional parallelepiped \( E_n \) spanned by the \( n \) column vectors \( v_j \) of \( A \).

Proof. Use the QR decomposition \( A = QR \), where \( Q \) is orthogonal and \( R \) is upper triangular. From \( Q^T Q = I \), we get \( 1 = \det(Q) \det(Q^T) = \det(Q^T) \) see that \( |\det(Q)| = 1 \). Therefore, \( \det(A) = \pm \det(R) \). The determinant of \( R \) is the product of the \( |R_{ij}| = |v_i - \text{proj}_{v_{i+1}} \cdots \text{proj}_{v_n} v_j| \) which was the distance from \( v_i \) to \( V_{j-1} \). The volume \( \det(E_{j-1}) \) of a \( j \)-dimensional checklist \( E_j \) with base \( E_{j-1} \) in \( V_{j-1} \) and height \( |u_j| \) is \( |\det(E_{j-1})| |u_j| \). Inductively \( \det(E_j) = |u_j| |\det(E_{j-1})| \) and therefore \( \det(E_n) = \prod_{i=1}^n |u_i| = \det(R) \).

The volume of a \( k \) dimensional parallelepiped defined by the vectors \( v_1, \ldots , v_k \) is \( \sqrt{\det(A^T A)} \).

Proof. \( Q^T Q = I_n \) gives \( A^T A = (QR)^T (QR) = R^T Q^T Q R = R^T R \). So, \( \det(R^T R) = \det(R)^2 = \prod_{k=1}^n |u_i|^2 \).

(Note that \( A \) is a \( n \times k \) matrix and that \( A^T A = R^T R \) and \( R \) is \( k \times k \) matrices.)

ORIENTATION. Determinants allow to define the orientation of \( n \) vectors in \( n \)-dimensional space. This is "handy" because there is no "right hand rule" in hyperspace... To do so, define the matrix \( A \) with column vectors \( v_j \) and define the orientation as the sign of \( \det(A) \). In three dimensions, this agrees with the right hand rule: if \( v_1 \) is the thumb, \( v_2 \) is the pointing finger and \( v_3 \) is the middle finger, then their orientation is positive.

\[ x , \det(A) = \]

CRAMER'S RULE. This is an explicit formula for the solution of \( A x = \bar{x} \). If \( A \) denotes the matrix, where the column \( v_i \) of \( A \) is replaced by \( \bar{b} \), then

\[ x_i = \det(A_i)/\det(A) \]

Proof. \( \det(A_i) = \det([v_1, \ldots , \bar{v}_i, \ldots , v_n]) = \det([v_1, \ldots , v_n]) \det(x_i) \). Thus \( x_i = \det([v_1, \ldots , \bar{v}_i, \ldots , v_n])/\det([v_1, \ldots , v_n]) \). Such a system with \( n \) equations and \( k \neq n \) unknowns, according to a short biography of Cramer by J.O. Conner and F. Robertson, the rule had however been used already before by other mathematicians. Solving systems with Cramer formulas is slower than by Gaussian elimination. The rule is still important. For example, if \( A \) or \( b \) depends on a parameter \( t \), and we want to see how \( x \) depends on the parameter \( t \) one can find explicit formulas for \( (d/dt)x(t) \).

\[ \begin{bmatrix} 5 & 3 & 2 \\ 8 & 5 & 3 \end{bmatrix} \]

and \( b = \begin{bmatrix} 8 \\ 2 \end{bmatrix} \) We get \( x = \begin{bmatrix} 8 \\ 3 \\ 2 \\ 5 \end{bmatrix} \) = 34y = \begin{bmatrix} 5 \\ 8 \\ 2 \end{bmatrix} = -54.

EXAMPLE. Solve the system \( 5x+3y = 8, 8x+5y = 2 \) using Cramer's rule. This linear system with \( A = \begin{bmatrix} 5 & 3 \\ 8 & 5 \end{bmatrix} \)

and \( b = \begin{bmatrix} 8 \\ 2 \end{bmatrix} \) We get \( x = \begin{bmatrix} 8 \\ 3 \\ 2 \\ 5 \end{bmatrix} \) = 34y = \begin{bmatrix} 5 \\ 8 \\ 2 \end{bmatrix} = -54.

EXAMPLES.\begin{bmatrix} 5 & 3 \\ 8 & 5 \end{bmatrix} \]

\[ \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 10 \end{bmatrix} \]

\[ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \\ 3 & 4 & 9 \end{bmatrix} \]

\[ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 4 & 2 \end{bmatrix} \]

\[ \begin{bmatrix} 1 & 6 & 10 & 1 \\ 2 & 8 & 17 & 1 \\ 0 & 0 & 3 & 8 \\ 0 & 0 & 0 & 5 \end{bmatrix} \]

1) \( A = \begin{bmatrix} 5 & 3 \\ 8 & 5 \end{bmatrix} \) Try row reduction. 2) \( A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \) Laplace expansion.

3) \( A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix} \) Partitioned matrix. 4) \( A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \) Make it triangular.

APPLYING HOFSTADTER BUTTERFLY. In solid state physics, one is interested in the function \( f(E) = \det(L - E I_n) \), where

\[ L = \begin{bmatrix} \cos(\alpha) & 1 & 0 & \cdots & 0 & 1 \\ \cos(2\alpha) & 1 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & \cos(\alpha) & \cdots & 0 \\ 0 & \cdots & 0 & \cos(3\alpha) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(n\alpha) \end{bmatrix} \]

describes an electron in a periodic crystal, \( E \) is the energy and \( \alpha = 2\pi/\lambda \). The electron can move as a Bloch wave whenever the determinant is negative. These intervals form the spectrum of the quantum mechanical system. A physicist is interested in the rate of change of \( f(E) \) or its dependence on \( \lambda \) when \( E \) is fixed.

The graph to the left shows the function \( E \rightarrow \log(|\det(L - E I_n)|) \) in the case \( \lambda = 2 \) and \( n = 5 \). In the energy intervals, where this function is zero, the electron can move, otherwise the crystal is an insulator. The picture to the right shows the spectrum of the crystal depending on \( \alpha \). It is called the "Hofstadter butterfly" made popular in the book "Gödel, Escher, Bach" by Douglas Hofstadter.

THE INVERSE OF A MATRIX. Because the columns of \( A^{-1} \) are solutions of \( A x = \bar{e}_i \), where \( \bar{e}_i \) is basis vectors, Cramers rule together with the Laplace expansion gives the formula:

\[ \left[ A^{-1} \right]_{ij} = (-1)^{i+j} \det(A_{ij})/\det(A) \]

\[ B_{ij} = (-1)^{i+j} \det(A_{ij}) \] is called the classical adjoint of \( A \). Note the change \( ij \rightarrow ji \). Don't confuse the classical adjoint with the transpose \( A^T \) which is sometimes also called the adjoint.