B-COORDINATES. Given a basis $\vec{v}_1, \ldots, \vec{v}_n$, define the matrix $S = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$. It is invertible. If $\vec{x} = \sum c_i \vec{v}_i$, then $c_i$ are called the B-coordinates of $\vec{x}$. We write $[\vec{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$. If $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$, we have $\vec{x} = S([\vec{x}]_B)$.

$B$-coordinates of $\vec{x}$ are obtained by applying $S^{-1}$ to the coordinates of the standard basis:

$[\vec{x}]_B = S^{-1}(\vec{x})$

EXAMPLE. If $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, then $S = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$. A vector $\vec{v} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$ has the coordinates

$S^{-1} \vec{v} = \begin{bmatrix} -5 \\ -3 \\ 6 \\ -9 \end{bmatrix}$

Indeed, as we can check, $-3\vec{v}_1 + 3\vec{v}_2 = \vec{v}$.

EXAMPLE. Let $V$ be the plane $x + y - z = 1$. Find a basis, in which every vector in the plane has the form $\begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$. SOLUTION. Find a basis, such that two vectors $\vec{v}_1, \vec{v}_2$ are in the plane and such that a third vector $\vec{v}_3$ is linearly independent to the first two. Since $(1, 0, 1), (0, 1, 1)$ are points in the plane and $(0, 0, 0)$ is in the plane, we can choose $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ which is perpendicular to the plane.

EXAMPLE. Find the coordinates of $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$ with respect to the basis $B = \{\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\}$. We have $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $S^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -1 & 0 \end{bmatrix}$. Therefore $[\vec{v}]_B = S^{-1} \vec{v} = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$. Indeed, $-\vec{v}_1 + 3\vec{v}_2 = \vec{v}$.

$B$-MATRIX. If $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis in $\mathbb{R}^n$ and $T$ is a linear transformation on $\mathbb{R}^n$, then the $B$-matrix of $T$ is defined as

$B = \begin{bmatrix} T(\vec{v}_1)_B & \cdots & T(\vec{v}_n)_B \end{bmatrix}$

COORDINATES HISTORY. Cartesian geometry was introduced by Fermat and Descartes (1596-1650) around 1636. It had a large influence on mathematics. Algebraic methods were introduced into geometry. The beginning of the vector concept came only later at the beginning of the 19th Century with the work of Bolzano (1781-1848). The full power of coordinates becomes possible if we allow to chose our coordinate system adapted to the situation. Descartes biography shows how far dedication to the teaching of mathematics can go ...