Problem 1) (20 points) True or False? No justifications are needed.

- T F Suppose $A$ is an $m \times n$ matrix, where $n < m$. If the rank of $A$ is $m$, then there is a vector $y \in \mathbb{R}^m$ for which the system $Ax = y$ has no solutions.
- T F The matrix \[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix}
\]
is invertible.
- T F The rank of an lower-triangular matrix equals the number of non-zero entries along the diagonal.
- T F The row reduced echelon form of a $3 \times 3$ matrix of rank 2 is one of the following:
\[
\begin{pmatrix}
1 & 0 & * \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
or
\[
\begin{pmatrix}
1 & * & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]
- T F The matrix \[
\begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}
\]
is a shear.
- T F For any matrix $A$, one has $\dim(\ker(A)) = \dim(\ker(\text{rref}(A)))$.
- T F If $\ker(A)$ is included in $\text{im}(A)$, then $A$ is not invertible.
- T F There exists an invertible $3 \times 3$ matrix, for which 7 of the 9 entries are $\pi$.
- T F The dimension of the image of a matrix $A$ is equal to the dimension of the image of the matrix $\text{rref}(A)$.
- T F There exists an invertible $n \times n$ matrix whose inverse has rank $n - 1$.
- T F If $A$ and $B$ are $n \times n$ matrices, then $AB$ is invertible if and only if both $A$ and $B$ are invertible.
- T F There exist matrices $A, B$ such that $A$ has rank 4 and $B$ has rank 7 and $AB$ has rank 5.
- T F There exist matrices $A, B$ such that $A$ has rank 2 and $B$ has rank 7 and $AB$ has rank 1.
- T F If for an invertible matrix $A$ one has $A^2 = A$, then $A = I_2$.
- T F If an invertible matrix $A$ satisfies $A^2 = 1$, then $A = I_2$ or $A = -I_2$.
- T F The matrix \[
\begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}
is invertible for every real number $c$.
- T F For $2 \times 2$ matrices $A$ and $B$, if $AB = 0$, then either $A = 0$ or $B = 0$.
- T F The determinant of a shear in the plane is always 1.
- T F The plane $x + y - z = 1$ is a linear subspace of three dimensional space.
- T F If $T$ is a rotation in space around an angle $\pi/6$ around the $z$ axes, then the linear transformation $S(x) = T(x) - x$ is invertible.

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Problem 2) (10 points)

Determine for each of the following matrices $A$, whether the system $A\vec{x} = \vec{e}_1$ has zero, one or infinitely many solutions and find the dimension of the image of $A$ in each case:

a) \[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 0 & 0 \\
1 & -2 & 0
\end{bmatrix}
\]
b) \[
\begin{bmatrix}
2 & 0 & 0 \\
0 & 2 & 2 \\
0 & 0 & 3
\end{bmatrix}
\]
c) \[
\begin{bmatrix}
5 & 0 & 0 \\
0 & 2 & 2 \\
0 & 2 & 2
\end{bmatrix}
\]
d) \[
\begin{bmatrix}
1 & 5 & 4 \\
0 & 1 & 3 \\
0 & 0 & -1
\end{bmatrix}
\]
e) \[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 0 \\
0 & 1 & 1
\end{bmatrix}
\]
f) \[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
g) \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

Solution:

a) Infinite. dim(im($A$)) = 2.
b) One. dim(im($A$)) = 3.
c) Infinite. dim(im($A$)) = 2
d) One. dim(im($A$)) = 3.
e) Infinite. dim(im($A$)) = 2.
f) Infinite. dim(im($A$)) = 1.
g) One. dim(im($A$)) = 3.

d) Find a matrix $B$ such that $B^2 = A$.

Problem 3) (10 points)

a) Write the matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ as a product of a rotation and a dilation.
b) What is the length of the vector $\vec{v} = A^{100}\vec{e}_1$, where $\vec{e}_1$ is the first basis vector?
c) In which direction does the vector $\vec{v}$ point?

Solution:

a) The matrix is a rotation dilation matrix, a rotation by $\pi/4$ and scaling by $\sqrt{2}$.
b) $A^{100}$ is a composition of a scaling by a factor $\sqrt{2}^{100} = 2^{50}$ and rotation by $\pi$. So, $A^{100} = \begin{bmatrix} -2^{50} & 0 & 0 \\ 0 & 0 & -2^{50} \end{bmatrix}$.
c) It points to $-\vec{e}_1$: after each 8 rotations, we are back to the initial position, so also after 96 rotations. The additional 4 rotations turn the vector to $-\vec{e}_1$.
d) A rotation by angle $\pi/8$ and scaling $2^{1/4}$ gives a rotation-dilation matrix with $a = 2^{1/4}\cos(\pi/8), b = 2^{1/4}\sin(\pi/8)$. The matrix is $A = 2^{1/4} \begin{bmatrix} \cos(\pi/8) & -\sin(\pi/8) \\ \sin(\pi/8) & \cos(\pi/8) \end{bmatrix}$.

Problem 4) (10 points)

Let $A$ be a $3 \times 3$ matrix such that $A^2 = 0$. That is, the product of $A$ with itself is the zero matrix.
a) Verify that Im($A$) is a subspace of ker($A$).
b) Can ran($A$) = 2? If yes, give an example.
c) Can ran($A$) = 1? If yes, give an example.
d) Can ran($A$) = 0? If yes, give an example.

Solution:

a) If $y$ is in the image, then $y = A(x)$ and $A(y) = A^2x = 0$.
b) No: If dim(ran($A$)) = 2, then dim(ker($A$)) = 1 and dim(ker($A^2$)) has maximal 2 dimensions so that the rank of $A^2$ would be at least 1.
c) Yes: $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
d) Yes: $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
Problem 5) (10 points)

Let $b, c$ be arbitrary numbers. Consider the matrix

$$A = \begin{bmatrix} 0 & -1 & b \\ 1 & 0 & -c \\ -b & c & 0 \end{bmatrix}.$$ 

a) Find $\text{rref}(A)$ and find a basis for the kernel and the image of $A$.

b) For which $b, c$ is the kernel one dimensional?

c) Can the kernel be two dimensional?

Solution:

a) $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & -b \\ 0 & 0 & 0 \end{bmatrix}.$

b) The kernel is always one-dimensional.

c) No. If the kernel had 2 or more dimensions, this would contradict the dimension formula $\dim(\ker(A)) + \dim(\text{im}(A)) = 3$.

Problem 6) (10 points)

Consider the matrix $A = \begin{bmatrix} 2 & 3 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 0 & 0 \end{bmatrix}$.

a) Use a series of elementary Gauss-Jordan row operations to find the reduced row echelon form $\text{rref}(A)$ of $A$. Do only one elementary operation at each step.

b) Find the rank of $A$.

c) Find a basis for the image of $A$.

d) Find a basis for the kernel of $A$.

Solution:

a) We end up with $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

b) The rank is 2, the number of pivot columns.

c) A basis of the image are the first two column vectors in $A$.

d) A basis of the kernel is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Problem 7) (10 points)

Let $A$ be a $2 \times 2$ matrix and $S = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$. We know that $B = S^{-1}AS = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find $A^{2003}$.

Hint. Write $B = (I_2 + C)$, note that $C^2 = 0$ and remember $(1 + x)^n = 1 + nx + \ldots + x^n$.

Solution:

$B^{2003} = (1 + 2003C)$ so that $A^{2003} = S(1 + 2003C)S^{-1} = 1 + 2003 (SCS^{-1})$. Because $S^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ and $2003SCS^{-1} = \begin{bmatrix} 6 & -4 \\ -9 & 6 \end{bmatrix}$ we have - would have been easier to add up 2000 year ago... - $A^{2003} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2003 \begin{bmatrix} 6 & 4 \\ -9 & -6 \end{bmatrix} = \begin{bmatrix} -12017 & 8012 \\ -18027 & 12019 \end{bmatrix}$. 

Problem 8) (10 points)

Let \( A \) be a \( 5 \times 5 \) matrix. Suppose a finite number of elementary row operations reduces \( A \) to
the following matrix
\[
B = \begin{bmatrix}
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
-1 & 0 & 0 & 1 & 1 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}.
\]

a) Find a basis of the kernel of \( A \).

b) Suppose the elementary row operations used in reducing \( A \) to \( B \) are the following:
   i) Add row 2 to row 3.
   ii) Swap row 2 and row 4.
   iii) Multiple row 4 by \( 1/2 \).
   iv) Subtract row 1 from row 5.

Find a basis of the image of \( A \).

Solution:

a) The matrix \( B \) and the matrix \( A \) have the same reduced row echelon form.
\[
\text{rref}(A) = \text{rref}(B) = B = \begin{bmatrix}
1 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]
This corresponds to \( v = 0, w = t, z = -2s \), so that \( v_1 = (1, 0, 0) \), \( v_2 = (0, -1/2, 1) \).

b) We reverse the steps:
   iv) Add row 1 to row 5
   iii) Multiply row 4 by \( 2 \)
   ii) Swap row 2 and row 4
   i) Subtract row 2 from row 3

\[
A = \begin{bmatrix}
-1 & 0 & 1 \\
0 & -1/2 & 1/2 \\
1 & 1 & -2
\end{bmatrix}.
\]

Pick columns 1,2,3,5 of the last matrix.

Problem 9) (10 points)

a) Find a basis for the plane \( x + 2y + z = 0 \) in \( \mathbb{R}^3 \).

b) Find a \( 3 \times 3 \) matrix which represents (with respect to the standard basis) a linear transformation with image the plane \( x + 2y + z = 0 \) and with the kernel the line \( x = y = z \).