ON SOLUTIONS OF LINEAR EQUATIONS

MATRIX. A rectangular array of numbers is called a **matrix**.

\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]

A matrix with \( m \) rows and \( n \) columns is called a \( m \times n \) matrix. A matrix with one column is a **column vector**. The entries of a matrix are denoted \( a_{ij} \), where \( i \) is the row number and \( j \) is the column number.

ROW AND COLUMN PICTURE. Two interpretations:

\[
A\vec{x} = \begin{pmatrix}
-w_1 \\
-w_2 \\
\vdots \\
w_m
\end{pmatrix} = \begin{pmatrix}
\vec{x}_1 \\
\vec{x}_2 \\
\vdots \\
\vec{x}_m
\end{pmatrix} = \begin{pmatrix}
\vec{v}_1 \\
\vec{v}_2 \\
\vdots \\
\vec{v}_m
\end{pmatrix}
\]

Row picture: each \( \vec{b}_i \) is the dot product of a row vector \( \vec{w}_i \) with \( \vec{x} \).

Column picture: \( \vec{b} \) is a sum of scaled column vectors \( \vec{v}_j \).

EXAMPLE. The system of linear equations

\[
\begin{align*}
3x - 4y - 5z &= 0 \\
x + 2y - z &= 0 \\
x - y + 3z &= 9
\end{align*}
\]

is equivalent to \( A\vec{x} = \vec{b} \), where \( A \) is a **coefficient matrix** and \( \vec{x} \) and \( \vec{b} \) are vectors.

\[
A = \begin{pmatrix}
3 & -4 & -5 \\
1 & 2 & -1 \\
-1 & -1 & 3
\end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}
\]

The augmented matrix (separators for clarity)

\[
B = \begin{pmatrix}
3 & -4 & -5 & 0 \\
1 & 2 & -1 & 0 \\
-1 & -1 & 3 & 9
\end{pmatrix}
\]

Row picture:

\[
0 = \begin{pmatrix} 3 & -4 & -5 \\ 1 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}
\]

Column picture:

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
3 \\
-4 \\
-5
\end{pmatrix} + \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
3 \\
-4 \\
-5
\end{pmatrix} + \begin{pmatrix}
3 \\
-4 \\
-5
\end{pmatrix} + \begin{pmatrix}
3 \\
-4 \\
-5
\end{pmatrix}
\]

SOLUTIONS OF LINEAR EQUATIONS. A system \( A\vec{x} = \vec{b} \) with \( m \) equations and \( n \) unknowns is defined by the \( m \times n \) matrix \( A \) and the vector \( \vec{b} \). The row reduced matrix \( \text{rref}(B) \) of \( B \) determines the number of solutions of the system \( A\vec{x} = \vec{b} \). There are three possibilities:

- **Consistent**; **Exactly one solution**. There is a leading 1 in each column of \( A \) but none in the last column of the augmented matrix \( B \).
- **Inconsistent**; **No solutions**. There is a leading 1 in the last row of \( B \).
- **Consistent**; **Infinitely many solutions**. There are columns in \( A \) without leading 1.

If \( m < n \) (less equations then unknowns), then there are either zero or infinitely many solutions.

The **rank** \( \text{rank}(A) \) of a matrix \( A \) is the number of leading ones in \( \text{rref}(A) \).

MURPHYS LAW.

"If anything can go wrong, it will go wrong."

"If you are feeling good, don’t worry, you will get over it!"

"Whenever you do Gauss-Jordan elimination, you screw up during the first couple of steps."

RELEVANCE OF EXCEPTIONAL CASES. There are important applications, where “unusual” situations happen: For example in medical tomography, systems of equations appear which are “ill posed”. In this case one has to be careful with the method.

The linear equations are then obtained from a method called the **Radon transform**. The task for finding a good method had led to a Nobel prize in Medicis 1979 for Allan Cormack. Cormack had sabbaticals at Harvard and probably has done part of his work on tomography here. Tomography helps today for example for cancer treatment.

MATRICES ALGEBRA. Matrices can be added, subtracted if they have the same size:

\[
A + B = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix} + \begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1n} \\
b_{21} & b_{22} & \cdots & b_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
b_{m1} & b_{m2} & \cdots & b_{mn}
\end{pmatrix} = \begin{pmatrix}
a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\
a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn}
\end{pmatrix}
\]

They can also be scaled by a scalar \( \lambda \):

\[
\lambda A = \begin{pmatrix}
\lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\
\lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn}
\end{pmatrix}
\]