LINEAR TRAFOS IN GEOMETRY

LINEAR TRANSFORMATIONS DEFORMING A BODY

A CHARACTERIZATION OF LINEAR TRANSFORMATIONS: a transformation $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ which satisfies $T(\vec{0}) = \vec{0}$, $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ and $T(\lambda \vec{x}) = \lambda T(\vec{x})$ is a linear transformation.

**Proof.** Call $\vec{v}_i = T(\vec{e}_i)$ and define $S(\vec{x}) = A\vec{x}$. Then $S(\vec{e}_i) = T(\vec{e}_i)$. With $\vec{x} = x_1 \vec{e}_1 + \ldots + x_n \vec{e}_n$, we have $T(\vec{x}) = T(x_1 \vec{e}_1 + \ldots + x_n \vec{e}_n) = x_1 \vec{v}_1 + \ldots + x_n \vec{v}_n$ as well as $S(\vec{x}) = A(x_1 \vec{e}_1 + \ldots + x_n \vec{e}_n) = x_1 \vec{v}_1 + \ldots + x_n \vec{v}_n$ proving $T(\vec{x}) = S(\vec{x}) = A\vec{x}$.

**SHEAR:**

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

In general, shears are transformations in the plane with the property that there is a vector $\vec{w}$ such that $T(\vec{w}) = \vec{w}$ and $T(\vec{x}) - \vec{x}$ is a multiple of $\vec{w}$ for all $\vec{x}$. If $\vec{u}$ is orthogonal to $\vec{w}$, then $T(\vec{x}) = \vec{x} + (\vec{u} \cdot \vec{x})\vec{w}$.

**SCALING:**

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

One can also look at transformations which scale $x$ differently than $y$ and where $A$ is a diagonal matrix.

**REFLECTION:**

$$A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$$

Any reflection at a line has the form of the matrix to the left. A reflection at a line containing a unit vector $\vec{u}$ is $T(\vec{x}) = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}$ with matrix $A = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$.

**PROJECTION:**

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

A projection onto a line containing unit vector $\vec{u}$ is $T(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u}$ with matrix $A = \begin{bmatrix} u_1u_1 & u_2u_1 \\ u_1u_2 & u_2u_2 \end{bmatrix}$. 
ROTATION:

\[ A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \]

Any rotation has the form of the matrix to the right.

ROTATION-DILATION:

\[ A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \quad A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \]

A rotation dilation is a composition of a rotation by angle \( \arctan(y/x) \) and a dilation by a factor \( \sqrt{x^2 + y^2} \). If \( z = x + iy \) and \( w = a + ib \) and \( T(x, y) = (X, Y) \), then \( X + iY = zw \). So a rotation dilation is tied to the process of the multiplication with a complex number.

BOOST:

\[ A = \begin{bmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{bmatrix} \]

The boost is a basic Lorentz transformation in special relativity. It acts on vectors \((x, ct)\), where \( t \) is time, \( c \) is the speed of light and \( x \) is space.

Unlike in Galileo transformation \((x, t) \mapsto (x + vt, t)\) (which is a shear), time \( t \) also changes during the transformation. The transformation has the effect that it changes length (Lorentz contraction). The angle \( \alpha \) is related to \( v \) by \( \tanh(\alpha) = \frac{v}{c} \). One can write also \( A(x, ct) = ((x + vt)/\gamma, t + (v/c^2)/\gamma x) \), with \( \gamma = \sqrt{1 - v^2/c^2} \).

ROTATION IN SPACE. Rotations in space are defined by an axes of rotation and an angle. A rotation by 120° around a line containing \((0, 0, 0)\) and \((1, 1, 1)\) belongs to \( A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \) which permutes \( \hat{e}_1 \rightarrow \hat{e}_2 \rightarrow \hat{e}_3 \).

REFLECTION AT PLANE. To a reflection at the \( xy \)-plane belongs the matrix \( A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \) as can be seen by looking at the images of \( \hat{e}_1 \). The picture to the right shows the textbook and reflections of it at two different mirrors.

PROJECTION ONTO SPACE. To project a 4d-object into xyz-space, use for example the matrix \( A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \). The picture shows the projection of the four dimensional cube (tesseract, hypercube) with 16 edges \((\pm 1, \pm 1, \pm 1, \pm 1)\). The tesseract is the theme of the horror movie "hypercube".