CONTINUOUS DYNAMICAL SYSTEMS.

A differential equation \( \frac{d}{dt} \mathbf{x} = f(\mathbf{x}) \) defines a dynamical system. The solutions is a curve \( \mathbf{x}(t) \) which has the velocity vector \( f(\mathbf{x}(t)) \) for all \( t \). We often write \( \dot{x} \) for \( \frac{d}{dt} x \).

ONE DIMENSION. A system \( \dot{x} = g(x,t) \) (written in the form \( \dot{x} = g(x,t), \dot{t} = 1 \)) and often has explicit solutions

- If \( \dot{x} = g(t) \), then \( x(t) = \int_0^t g(s) \, ds \).
- If \( \dot{x} = h(x) \), then \( dx/h(x) = dt \) and so \( t = \int_0^x h(x) \, dx = \int_0^t g(t) \, dt = G(t) \) so that \( x(t) = H^{-1}(G(t)) \).
- If \( \dot{x} = g(t)/h(x) \), then \( H(x) = \int_0^x h(x) \, dx = \int_0^{\dot{x}} g(t) \, dt = G(t) \) so that \( x(t) = H^{-1}(G(t)) \).

In general, we have no closed form solutions in terms of known functions. The solution \( x(t) = \int_0^t e^{-t^2} \, dt \) of \( \dot{x} = e^{-t^2} \) for example can not be expressed in terms of functions exp, sin, log, \( \sqrt{\cdot} \) etc but it can be solved using Taylor series: because \( e^{-t^2} = 1 - t^2 + t^4/2! - t^6/3! + \ldots \) taking coefficient wise the anti-derivatives gives:

\[
x(t) = t - t^3/3 + t^5/(3!2) - t^7/(7!3) + \ldots
\]

HIGHER DIMENSIONS. In higher dimensions, chaos can set in and the dynamical system can become unpredictable.

The nonlinear Lorentz system to the right

\[
\begin{align*}
\dot{x}(t) &= 10(y(t) - x(t)), \\
\dot{y}(t) &= -x(t)z(t) + 28x(t) - y(t), \\
\dot{z}(t) &= x(t)z(t) - 8z(t) + 3.
\end{align*}
\]

shows a “strange attractor”. Even so completely deterministic: (from \( x(0) \) all the path \( x(t) \) is determined), there are observables which can be used as a random number generator. The Duffing system \( \ddot{x} + \dot{x} + x + x^3 - 12 \cos(t) = 0 \) to the left can be written in the form \( \mathbf{v} = \mathbf{f}(\mathbf{v}) \) with a vector \( \mathbf{v} = (x, \dot{x}, t) \).

1D LINEAR DIFFERENTIAL EQUATIONS. A linear differential equation in one dimension \( \dot{x} = \lambda x \) has the solution \( x(t) = e^{\lambda t} x(0) \). This differential equation appears

- as population models with \( \lambda > 0 \): birth rate of the population is proportional to its size.
- as a model for radioactive decay with \( \lambda < 0 \): the rate of decay is proportional to the number of atoms.

LINEAR DIFFERENTIAL EQUATIONS IN HIGHER DIMENSIONS. Linear dynamical systems have the form \( \dot{x} = Ax \), where \( A \) is a matrix. Note that the origin \( \mathbf{0} \) is an equilibrium point: if \( x(0) = 0 \), then \( x(t) = 0 \) for all \( t \). The general solution is \( x(t) = e^{At} = 1 + At + A^2 t^2/2! + \ldots \) because \( \dot{x}(t) = A + 2A^2 t/2! + \ldots = A(1 + At + A^2 t^2/2! + \ldots) = Ae^{At} = Ax(t) \).

If \( B = S^{-1}AS \) is diagonal with the eigenvalues \( \lambda_j = a_j + ib_j \) in the diagonal, then \( y = S^{-1}x \) satisfies \( \dot{y}(t) = e^{Bt} \) and therefore \( y_j(t) = e^{\lambda_j t} y_j(0) = e^{a_j t} e^{ib_j t} y_j(0) \). The solutions in the original coordinates are \( x(t) = S y(t) \).

PHASE PORTRAITS. For differential equations \( \dot{x} = f(x) \) in 2D one can draw the vector field \( x \rightarrow f(x) \). The solution \( x(t) \) is tangent to the vector \( f(x(t)) \) everywhere. The phase portraits together with some solution curves reveal much about the system. Some examples of phase portraits of linear two-dimensional systems.
UNDERSTANDING A DIFFERENTIAL EQUATION. The closed form solution like $x(t) = e^{At}x(0)$ for $\dot{x} = Ax$ is actually quite useless. One wants to understand the solution quantitatively. Questions one wants to answer are: what happens in the long term? Is the origin stable, are there periodic solutions. Can one decompose the system into simpler subsystems? We will see that diagonalisation allows to understand the system: by decomposing it into one-dimensional linear systems, which can be analyzed seperately. In general, "understanding" can mean different things:

- Plotting phase portraits.
- Computing solutions numerically and estimate the error.
- Finding special solutions.
- Predicting the shape of some orbits.
- Finding regions which are invariant.

Finding special closed form solutions $x(t)$. Finding a power series $x(t) = \sum_n a_n t^n$ in $t$. Finding quantities which are unchanged along the flow (called "Integrals"). Finding quantities which increase along the flow (called "Lyapunov functions").

LINEAR STABILITY. A linear dynamical system $\dot{x} = Ax$ with diagonalizable $A$ is linearly stable if and only if $a_j = \text{Re}(\lambda_j) < 0$ for all eigenvalues $\lambda_j$ of $A$.

PROOF. We see that from the explicit solutions $y_j(t) = e^{a_jt} e^{ib_jt} y_j(0)$ in the basis consisting of eigenvectors. Now, $y(t) \to 0$ if and only if $a_j < 0$ for all $j$ and $x(t) = Sy(t) \to 0$ if and only if $y(t) \to 0$.

RELATION WITH DISCRETE TIME SYSTEMS. From $\dot{x} = Ax$, we obtain $x(t + 1) = Bx(t)$, with the matrix $B = e^A$. The eigenvalues of $B$ are $\mu_j = e^{\lambda_j}$. Now $|\mu_j| < 1$ if and only if $\text{Re}\lambda_j < 0$. The criterium for linear stability of discrete dynamical systems is compatible with the criterium for linear stability of $\dot{x} = Ax$.

EXAMPLE 1. The system $\dot{x} = y, \dot{y} = -x$ can in vector form $v = (x, y)$ be written as $\dot{v} = Av$, with $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The matrix $A$ has the eigenvalues $i, -i$. After a coordinate transformation $w = S^{-1}v$ we get with $w = (a, b)$ the differential equations $\dot{a} = ia, \dot{b} = -i = b$ which has the solutions $a(t) = e^{it}a(0), b(t) = e^{-it}b(0)$. The original coordinates satisfy $x(t) = \cos(t)x(0) - \sin(t)y(0), \quad y(t) = \sin(t)x(0) + \cos(t)y(0)$. Indeed $e^{At}$ is a rotation in the plane.

EXAMPLE 2. A harmonic oscillator $\ddot{x} = -x$ can be written with $y = \dot{x}$ as $\dot{x} = y, \dot{y} = -x$ (see Example 1). The general solution is therefore $x(t) = \cos(t)x(0) - \sin(t)\dot{x}(0)$.

EXAMPLE 3. We take two harmonic oscillators and couple them: $\ddot{x}_1 = -x_1 - \epsilon(x_2 - x_1), \ddot{x}_2 = -x_2 + \epsilon(x_2 - x_1)$. For small $x_i$ one can simulate this with two coupled penduli. The system can be written as $\ddot{v} = Av$, with $A = \begin{bmatrix} -1 + \epsilon & -\epsilon \\ -\epsilon & -1 + \epsilon \end{bmatrix}$. The matrix $A$ has an eigenvalue $\lambda_1 = -1$ to the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and an eigenvalue $\lambda_2 = -1 + 2*\epsilon$ to the eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The coordinate change $S$ is $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. It has the inverse $S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}/2$. In the coordinates $w = S^{-1}v = [y_1, y_2]$, we have oscillations $\ddot{y}_1 = -y_1$, corresponding to the case $x_1 - x_2 = 0$ (the pendula swing synchronously) and $\ddot{y}_2 = -(1 - 2\epsilon)y_2$ corresponding to $x_1 + x_2 = 0$ (the pendula swing against each other).