Mathematics 21b

Final Exam
May 22, 2001

Your Section (circle one):

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The exam will last 3 hours.
No calculators are allowed.
Justify your answers carefully (except in Questions 1, 2 and 3).
Write your final answers in the spaces provided.
(1) True or False (no explanation is necessary).

T  F : If $A$ and $B$ are 2 $\times$ 2 rotation-dilation matrices then $AB = BA$.

T  F : There is a 3 $\times$ 5 matrix $A$ and a 5 $\times$ 3 matrix $B$ with $AB = I_3$ and $BA = 0$ (the zero matrix).

T  F : All shears have the same determinant.

T  F : If $A$ is a symmetric matrix then the kernel of $A$ is the orthogonal complement of the image of $A$.

T  F : For any matrix $A$, the product $AA^T$ is diagonalisable.
T  F : If $A$ and $B$ are $n \times n$ matrices with $A$ invertible then $B$ and $ABA^{-1}$ have the same eigenvectors.

T  F : If all the eigenvalues of $A$ have absolute value less than 1 then $\bar{0}$ is a stable equilibrium for the linear system $\bar{x}(t + 1) = A\bar{x}(t)$.

T  F : If all the eigenvalues of $A$ have absolute value less than 1 then $\bar{0}$ is a stable equilibrium for the linear system $d\bar{x}(t)/dt = A\bar{x}(t)$.

T  F : If $T : C^\infty \rightarrow C^\infty$ is a linear transformation with image $C^\infty$ then $T$ is invertible (in the sense that there is a linear transformation $S : C^\infty \rightarrow C^\infty$ with both $ST$ and $TS$ equal to the identity).

T  F : If $T : C^\infty \rightarrow C^\infty$ is a linear transformation with kernel $\{0\}$ then $T$ is invertible (in the sense that there is a linear transformation $S : C^\infty \rightarrow C^\infty$ with both $ST$ and $TS$ equal to the identity).
[This page is blank for rough working.]
(2) For each of the following matrices and linear transformations circle the value of its determinant. (No explanation is necessary.)

\[-1 \ 0 \ 1 : \begin{bmatrix} 1 & 0 & 1 \\ 4 & 1 & 2 \\ 3 & 0 & 3 \end{bmatrix} \]

\[-1 \ 0 \ 1 : \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 \\ 1 & 2 & 3 & 5 \\ 1 & 2 & 4 & 4 \end{bmatrix} \]

\[-1 \ 0 \ 1 : \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 3 & 2 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[-1 \ 0 \ 1 : \text{Reflection in the plane } x + 2y + 3z = 0 \text{ in } \mathbb{R}^3.\]

\[-1 \ 0 \ 1 : \text{Orthogonal projection to the plane } x + 2y + 3z = 0 \text{ in } \mathbb{R}^3.\]
(3) Below we have five differential equations and five graphs. Next to each differential equation write the number of the graph that represents a solution to that differential equation. (No explanation is necessary.)

(a) $df/dt + f = 3e^{2t}$ has a solution represented by graph:

(b) $d^2f/dt^2 - f = 0$ has a solution represented by graph:

(c) $d^2f/dt^2 + f = 0$ has a solution represented by graph:

(d) $16d^2f/dt^2 - 8df/dt + 17f = -16 \cos 2t - 47 \sin 2t$ has a solution represented by graph:

(e) $d^2f/dt^2 + f = 2 \cos t$ has a solution represented by graph:
[This page is blank for rough working.]
(4) Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ denote rotation through $120^\circ$ about the line $x = y = z$ (in a clockwise direction when facing the origin from $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$), let $S : \mathbb{R}^3 \to \mathbb{R}^3$ denote reflection in the plane $x = y$, and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ denote reflection in the plane $y + z = 0$.

(a) Find the matrices for $R$, $S$ and $T$ (with respect to the standard basis of $\mathbb{R}^3$).

(b) Find the matrices for $RS$, $ST$ and $RST$ (with respect to the standard basis of $\mathbb{R}^3$).
(c) Give a (precise) geometric description of each of the linear transformations $RS$, $ST$ and $RST$. 
(5) The matrix

\[ B = \begin{bmatrix}
2 & 2 & 2 & 0 & 6 \\
-1 & -1 & -1 & 1 & 0 \\
2 & 2 & 0 & 2 & 8 \\
-1 & -1 & 0 & 1 & 2
\end{bmatrix} \]

has rank 3. Let \( E \) denote the \( 4 \times 9 \) augmented matrix \((B|I_4)\).

(a) Find \( \text{rref } E \).
(b) Find a basis for the image of $B$ and a basis for the kernel of $B$. 
(c) Find a $5 \times 3$ matrix $D$ such that

$$(\text{ref } B)D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(d) Find a $3 \times 4$ matrix $A$ and a $5 \times 3$ matrix $C$ such that $ABC = I_3$. 
(6) The matrix \[ A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \] has eigenvalues \(-1\) and \(3\).

(a) Find an orthonormal eigenbasis of \(\mathbb{R}^4\) for \(A\).
[Space to continue part (a).]

(b) Find an orthogonal matrix \( S \) and a diagonal matrix \( D \) such that \( AS = SD \).
(c) Calculate $A^{100}$. 
(7) Consider the nonlinear system

\[ \begin{align*}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= x^2 - 2x - y.
\end{align*} \]

(a) Sketch the nullclines and a rough direction field for this system.
(b) What are the equilibrium points? Near each equilibrium point linearise the equations, determine the approximate behaviour and state whether or not that equilibrium point is stable.
(c) Sketch a possible phase portrait.
(8) (a) Find the Fourier series for the function \( f \in C[-\pi, \pi] \) which equals \( x(x - \pi) \) in the region \([0, \pi]\) and equals \(-x(x + \pi)\) in the region \([-\pi, 0]\).

[HINT: An indefinite integral of \( x(x - \pi) \sin nx \) is
\[
(x(-x)/n) \cos nx + ((2x - \pi)/n^2) \sin nx + (2/n^3) \cos nx.
\]
(b) Compute

$$\sum_{m=0}^{\infty} (-1)^m / (2m + 1)^3 = 1 - 1/3^3 + 1/5^3 - 1/7^3 + 1/9^3 - \ldots$$
(c) Solve the equation
\[ \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \]
in the region \(0 \leq x \leq \pi, t \geq 0\) subject to \(T(0,t) = 0, T(\pi,t) = \pi^2\) and \(T(x,0) = x^2\). [HINT: \(\pi x\) satisfies all but the last of these conditions.]