8. The ends of a copper wire of length \( \pi \) are heated so that their temperatures are \( T(t, 0) = t \) and \( T(t, \pi) = t + \frac{\pi t}{2} \), respectively. The temperature of the wire at time \( t = 0 \) is given by \( T(0, x) = \sin x + \frac{x^2}{2} \). Assuming that \( T(t, x) \) satisfies the heat equation

\[
\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}
\]

the special case where \( \mu = 1 \), find \( T(t, x) \) for all times \( t \geq 0 \) and all points on the wire \( 0 \leq x \leq \pi \).

Hint: Find a particular solution \( S(t, x) \) to the heat equation such that \( T(t, 0) - S(t, 0) = 0 \) and \( T(t, \pi) - S(t, \pi) = 0 \).

There is such a solution \( S(t, x) \) which is a polynomial in the variables \( t \) and \( x \). Then use the linearity of solutions of the heat equation (the superposition principle) to find \( T(t, x) \).

Solution: Let's find a suitable \( S(t, x) \). We have the guess that \( S(t, x) = \frac{x^2}{2} + f(t) \) for some \( f \).
We want $S$ to solve the heat equation.

$$\frac{\partial S}{\partial t} = f'(t), \quad \frac{\partial^2 S}{\partial x^2} = 1, \quad \text{so } f'(t) \equiv 1.$$ Let's choose $f(t) \equiv t,$ so $S(t, x) = t + \frac{x^2}{2}.$

That $S$ are solutions to the heat equation, so by linearity $T - S$ is too. Define $U(t, x) = T(t, x) - S(t, x).

Now  $U(t, 0) = T(t, 0) - S(t, 0) = t - t = 0.$

$U(t, \pi) = T(t, \pi) - S(t, \pi) = t + \frac{\pi^2}{2} - (t + \frac{\pi^2}{2}) = 0$.

$U(0, x) = T(0, x) - S(0, x) = (\sin x + \frac{x^2}{2}) - (\frac{x^2}{2} + \sin x).$

Remember that the *typical* solution $V$ to the heat equation with boundary condition $V(t, 0) = V(t, \pi) = 0$ looks like

$$U(t, x) = \sum_{n=1}^{\infty} A_n \sin nx \ e^{-n^2 t}.$$ We want $U(t, x)$ to be of this form; we can take $A_1 = 1$ as all other $A_n = 0$.

So $U(t, x) = \sin x \ e^{-t}.$

Now $U = T - S,$ so $T = U + S,$ so $T(t, x) = e^{-t} \sin x + t + \frac{x^2}{2}.$