Content

Preliminaries

Discrete dynamical systems

Differential equations

Fourier analysis

Partial differential equations
Organisation of this lecture

The *New York Times* recently ran an article titled [PowerPoint Makes You Dumb](http://www.nytimes.com). In it, the author cites sources who blame the Columbia space shuttle disaster on Microsoft’s presentation software. You see, apparently those guys at NASA used a PowerPoint slide to “explain” something crucial.

But after spending several hours designing a mock slide show, Burne realises that it is the PowerPoint interface itself that is the problem. PowerPoint forces us to think linearly, and that can lead to poor decisions. The slides are too easy to create, and too easy to update, so we don’t think about the content. It’s the PowerPoint, stupid!

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*Less Tangible*

*It's the PowerPoint, Stupid*
I) Preliminaries

- Diagonalization
- Complex Numbers
- Linear spaces, linear transformations
- Differential operators
Diagonalization

Diagonalization of a matrix $A$ is possible if:

- $A$ is a symmetric matrix
- All eigenvalues of $A$ are different

Prototype of nondiagonalizable matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
Jordan Normal Form

\[
\begin{bmatrix}
\lambda & 1 & 0 & 0 & 0 & 0 \\
0 & \lambda & 1 & 0 & 0 & 0 \\
0 & 0 & \lambda & 1 & 0 \\
0 & 0 & 0 & \lambda & 1 \\
0 & 0 & 0 & 0 & \lambda \\
0 & 0 & 0 & 0 & 0 & \lambda
\end{bmatrix}
\]
Complex Numbers

\[ Z = x + iy = r \exp(i \theta) \]
Fundamental Theorem of algebra

A polynomial of degree $n$ has exactly $n$ roots.

- If $a+ib$ is a root, then $a-ib$ is a root too.
- If $n$ is odd, there is at least one real root.
Roots of One

Example:

Permutation matrix:

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[p(\lambda) = (\lambda^n - 1) = (\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_n)\]
Linear Spaces

In a linear space, we can add and scale.

\[
\begin{array}{c|c}
\mathbb{R}^n & \text{M}(\mathbb{R}, n) \\
C^\infty(\mathbb{R}) & C^\infty([\pi, \pi])
\end{array}
\]

To check whether a subset \( X \) of one of these spaces is a linear space, we check:

- \( x+y \) is in \( X \)
- \( rx \) is in \( X \)
- \( 0 \) is in \( X \)
Linear Maps

To check whether a map between linear spaces is linear, we have to check:

\[ T(x+y) = T(x) + T(y), \quad T(rx) = r \cdot T(x), \quad T(0) = 0 \]
Differential operators

\[ Df = f' \]

\( p(x) \) polynomial

\( T = p(D) \) differential operator

\( T f = g \) differential equation

Fundamental theorem of calculus:

\[
p(D) = (D - \lambda_1)(D - \lambda_2)...(D - \lambda_n)
\]
This is a lot of material!
“We like it extreme”
Quizz coming up!

Structure of the quizz:

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Shout: 21 14
Linear space or not?

2) Smooth $2\pi$-periodic functions with $\int_0^{2\pi} f(x) \, dx = 0$.
3) \( \{ f \in C^\infty(\mathbb{R}) \mid f(10) = 1 \} \)
5) Smooth $2\pi$-periodic functions satisfying $f'(0) = 0$.
7) $2 \times 2$ matrices satisfying $\text{tr}(A) = 0$.
11) $2 \times 2$ matrices satisfying $\det(A) = 0$.

Linear map or not?

2) $T(f)(x) = x^2 f(x)$.
3) $T(f)(x) = f(1)^2 + f(x)$.
5) $T(f)(x) = f'(x)$.
7) $T(f)(x) = f(x)f'(x)$.
11) $T(f)(x) = x + f(x)$.
II) Discrete Dynamical Systems

- Solving initial value problems
- Analyse phase space
- Find out about stability
Asymptotic Stability

\[ A\vec{x} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/2 \end{bmatrix} \vec{x} \]
Hyperbolic Behavior

\[ A\vec{x} = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix} \vec{x} \]
Expansive behavior

\[ A\vec{x} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \vec{x} \]
Rotational Behavior

\[ A \vec{x} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \vec{x} \]
Rotation Dilation

\[
A\vec{x} = \frac{1}{2} \begin{bmatrix}
\cos(\alpha) & \sin(\alpha) \\
-\sin(\alpha) & \cos(\alpha)
\end{bmatrix} \vec{x}
\]
Rotation Dilation

\[ A\vec{x} = 2 \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \vec{x} \]
Eigenvalues

The eigenvalues of A determine the stability of the origin.

Asymptotic stability

All eigenvalues have absolute value $|\lambda| < 1$
Stability for discrete systems

\[ x(t+1) = Ax \]

Determinant

\[ |\text{tr}(A)| - 1 < \det(A) < 1 \]
Initial Value Problem

- Diagonalize A
- Write $x$ as sum of eigenvector
- Write down solution

Example Blackboard
III) Differential Equations

- Solving
- Phase space
- Stability
Differential equations

\[
\begin{align*}
\dot{x} &= f(x, y) \\
\dot{y} &= g(x, y)
\end{align*}
\]
Linear Systems

\[ \begin{align*}
\dot{x} &= ax + by \\
\dot{y} &= cx + dy
\end{align*} \]

\[ A = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \]

The eigenvalues of A determine the dynamical behavior.
ID Case

\[
\frac{d}{dt} x = \lambda x \\
x(t) = e^{\lambda t} x(0)
\]
Real, nonzero eigenvalues
Complex eigenvalues
Some zero eigenvalue
Asymptotic Stability
Stability for differential equation

\[ \dot{x} = Ax \]

Determinant

Trace
Stability Examples

Example Blackboard
III) Nonlinear differential equation

- Equilibrium points
- Nullclines
- Nature of equilibrium points
- Understanding phase space
Nonlinear Systems

\[
\dot{x} = f(x, y) \\
\dot{y} = g(x, y)
\]
Jacobean matrix

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$$

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_x(x, y) & f_y(x, y) \\ g_x(x, y) & g_y(x, y) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
Intermezzo: pendulum
Example

Example Blackboard
IV) Higher Order Differential Equations

- Solving initial value problem $p(D)f = g$
- Homogenous problem $p(D)f = 0$
- Finding special solution to inhomogenous problem.
Higher Order ODE's

\[ \frac{d}{dt} x(t) = Ax(t) \]

Important special case, which can be reduced to the above matrix notation are nonhomogenous higher order differential equations:

\[ p(D) f(x) = g(x) \]
Initial Value Problem

Problem:
\[ f'' + 5f = 2 \sin(3x) \]
\[ f(0) = 1, \quad f'(0) = 2 \]

**Homogeneous**
\[ f'' + 5f = 0 \]
\[ \lambda^2 + 5\lambda = 0 \]
\[ \lambda = i\sqrt{5}, -i\sqrt{5} \]
\[ f(x) = a \sin(\sqrt{5}x) + b \cos(\sqrt{5}x) \]

**Inhomogeneous**
\[ f'' + 5f = 2 \sin(3x) \]
\[ f(x) = c \sin(3x) \]
\[ f'' + 5f = c + 5c = 2 \]
so \( c = -\frac{1}{2} \)
\[ f(x) = -\sin(3x)/2 \]

Initial conditions
\[ f(0) = 1 \] implies \( b = 1 \)
\[ f'(0) = 1 \] implies \( a = -\frac{3}{2\sqrt{5}} \)

Solution:
\[ f(x) = -\frac{3}{2\sqrt{5}} \sin(\sqrt{5}x) + \cos(\sqrt{5}x) - \sin(3x)/2 \]
Example

Example Blackboard
V) Fourier analysis

- Fourier Series
- Symmetry
- Parceval
Fourier series

\[ f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n \geq 1} a_n \cos(nx) + b_n \sin(nx) \]
Fourier coefficients

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \frac{1}{\sqrt{2}} \, dx \]

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx \]
Sound synthesis I
Sound synthesis II
Sound synthesis III
Fourier approximation
Even and Odd

cos-series

sin-series
Parseval

\[ f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \]

\[ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \|f\|^2 \]
Example

Example Blackboard
VI) Partial Differential equations

- Heat equation
- Wave equation
- Laplace equation (not covered)
Heat Evolution
Waves
Wave Evolution
Example

Example Blackboard
Pro Memoria

- Exam is on January 27
- Review old exams and homework
- Do the practice exam