VECTOR FIELDS:

Planar vector field.
A vector field in the plane is a map, which assigns to each point \((x, y)\) in the plane a vector \(F(x, y) = (P(x, y), Q(x, y))\).

Vector field in space.
A vector field in space is a map, which assigns to each point \((x, y, z)\) in space a vector \(F(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))\).

Planar vector field examples.
1) \(F(x, y) = (y, -x)\) is a planar vector field which you see in a picture on the right.
2) \(F(x, y) = (x-1, y)/(x-1)^2+y^2)^{1/2} - (x+1, y)/(x+1)^2+y^2)^{1/2}\) is the electric field of positive and negative point charge. It is called dipole field. It is shown in the picture above.

Gradient field.
If \(f(x, y)\) is a function of two variables, \(\nabla f = (\partial f/\partial x, \partial f/\partial y)\), is a vector field which is normal to \(\partial f/\partial x \neq 0\). The same in 3D: gradient fields are of the form \(\nabla f = (\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)\).

Example of a vector field. if \(H(x, y) = x^2 + y^2\) is a Hamiltonian vector field. An example is the harmonic oscillator \(H(x, y) = x^2 + y^2\). Its vector field \((H_x(x, y), -H_y(x, y)) = (y, -x)\) is the same as in example 1) above.

When is a vector field a gradient field (2D)?
\[ F(x, y) = (P(x, y), Q(x, y)) = \nabla f(x, y) \]
implies \(Q_x(x, y) = P_y(x, y)\). If this does not hold at some point, \(F\) is no gradient field. We will see next week that the condition \(\text{curl}(F) = Q_x - P_y = 0\) is also necessary for \(F\) to be a gradient field.

Example: Vector fields in biology.
Let \(x(t)\) denote the population of a "prey species" like tuna fish and \(y(t)\) is the population size of a "predator" like sharks. We have \(x'(t) = ax(t) + bx(t)y(t)\) with positive \(a, b\) because both more predators and more prey species will lead to prey consumption. The rate of change of \(y(t)\) is \(-cy(t) + dxy\), where \(c, d\) are positive.

Newton’s law \(m\ddot{x} = F\) relates the acceleration \(\ddot{x}\) of a body with the force \(F\) acting on it at the point. For example, if \(x(t)\) is the position of a mass point in \([-1, 1]\) attached to two springs and the mass is \(m = 2\), then the point experiences a force \((-x + -x) = -2x\) so that \(m\ddot{x} = 2x\) or \(x'(t) = -x(t)\). If we introduce \(y(t) = x'(t)\) of \(t\), then \(x'(t) = y(t)\) and \(y'(t) = -x(t)\). Of course \(y\) is the velocity of the mass point, so a pair \((x, y)\), though of as an initial condition, describes the system so that nature knows what the future evolution of the system has to be given that data.

We don’t yet know yet the curve \(t \mapsto (x(t), y(t))\), but we know the tangents \(x'(t), y'(t)) = ((y(t), -x(t))\). In other words, we know a direction at each point. The equation \(x'(t) = y(t)\) is called a system of ordinary differential equations (ODE). More generally, the problem when studying ODE’s is to find solutions \(x(t), y(t)\) of equations \(x'(t) = f(x(t), y(t)), y'(t) = g(x(t), y(t))\).

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Hamiltonian vector field.
An example is the harmonic oscillator \(H(x, y) = x^2 + y^2\). Its vector field \((H_x(x, y), -H_y(x, y)) = (y, -x)\) is the same as in example 1) above.

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We have a negative sign in the first part because predators would die out without food. The second term is explained because both more predators as well as more prey leads to a growth of predators through reproduction.

A concrete example is the Volterra-Lodka system
\[
\begin{align*}
\dot{x} &= 0.4x - 0.4xy \\
\dot{y} &= -0.1y + 0.2xy
\end{align*}
\]
Volterra explained with such systems the oscillation of fish populations in the Mediterranean sea. At any specific point \((x, y) = (x(t), y(t))\), there is a curve \(\vec{r}(t) = (x(t), y(t))\) through that point for which the tangent \(\vec{r}'(t) = (x'(t), y'(t))\) is the vector \((0.4x - 0.4xy, -0.1y + 0.2xy)\).

Newton’s law \(m\ddot{x} = F\) relates the acceleration \(\ddot{x}\) of a body with the force \(F\) acting at the point. For example, if \(x(t)\) is the position of a mass point in \([-1, 1]\) attached to two springs and the mass is \(m = 2\), then the point experiences a force \((-x + -x) = -2x\) so that \(m\ddot{x} = 2x\) or \(x'(t) = -x(t)\). If we introduce \(y(t) = x'(t)\) of \(t\), then \(x'(t) = y(t)\) and \(y'(t) = -x(t)\). Of course \(y\) is the velocity of the mass point, so a pair \((x, y)\), thought of as an initial condition, describes the system so that nature knows what the future evolution of the system has to be given that data.

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Vector fields in meteorology.
On maps like http://www.hpc.ncep.noaa.gov/sfc/satsfc.gif one can see isotherms, curves of constant temperature or pressure \(p(x, y) = c\). These are level curves. The wind maps are vector fields. \(F(x, y)\) is the wind velocity at the point \((x, y)\). The wind velocity \(F\) is not always normal to the isobars, the lines of equal pressure \(p\). The scalar pressure field \(p\) and the velocity field \(F\) depend on time. The equations which describe the weather dynamics are called the Navier Stokes equations
\[
\frac{d}{dt}F + F \cdot \nabla F = \nabla p + \nabla \times (\eta \nabla u) = 0
\]
(we will see what is \(\Delta, \nabla \) later.) It is a partial differential equation like \(u_t - \nabla u = 0\). Finding solutions is not trivial: 1 Million dollars are given to the person proving that the equations have smooth solutions in space.